MA3303 PROBABILITY AND COMPLEX FUNCTIONS

UNIT I PROBABILITY AND RANDOM VARIABLES (12- Periods)

Axioms of probability – Conditional probability – Baye's theorem – Discrete and continuous random variables – Moments – Moment generating functions – Binomial, Poisson, Geometric, Uniform, Exponential and Normal distributions – Functions of a random variable.

UNIT II TWO-DIMENSIONAL RANDOM VARIABLES (12- Periods)

Joint distributions – Marginal and conditional distributions – Covariance – Correlation and linear regression – Transformation of random variables – Central limit theorem (for independent and identically distributed random variables).

UNIT III ANALYTIC FUNCTIONS (12- Periods)

Analytic functions – Necessary and sufficient conditions for analyticity in Cartesian and polar coordinates – Properties – Harmonic conjugates – Construction of analytic function – Conformal mapping – Mapping by functions – Bilinear transformation.

UNIT IV COMPLEX INTEGRATION (12- Periods)

Line integral – Cauchy's integral theorem – Cauchy's integral formula – Taylor's and Laurent's series– Singularities – Residues – Residue theorem – Application of residue theorem for evaluation of real integrals – Applications of circular contour and semicircular contour (with poles NOT on real axis).

UNIT V ORDINARY DIFFERENTIAL EQUATIONS (12- Periods)

Higher order linear differential equations with constant coefficients – Method of variation of parameters– Homogenous equation of Euler's and Legendre's type – System of simultaneous linear first order differential equations with constant coefficients – Method of undetermined coefficients.

TEXT BOOKS

1. Johnson. R.A., Miller. I and Freund. J., "Miller and Freund's Probability and Statistics for Engineers", Pearson Education, Asia, 9th Edition, 2016.
2. Milton. J. S. and Arnold. J.C., "Introduction to Probability and Statistics", Tata McGraw Hill, 4thEdition, 2007.
3. Grewal.B.S., "Higher Engineering Mathematics", Khanna Publishers, New Delhi, 44th Edition,2018.

REFERENCES

1. Devore. J.L., "Probability and Statistics for Engineering and the Sciences", Cengage Learning, New Delhi, 8th Edition, 2014.
2. Papoulis. A. and Unnikrishnapillai . S., "Probability, Random Variables and Stochastic Processes", McGraw Hill Education India, 4th Edition, New Delhi, 2010.
3. Ross . S.M., "Introduction to Probability and Statistics for Engineers and Scientists", 5thEdition, Elsevier, 2014.
4. Spiegel. M.R., Schiller. J. and Srinivasan . R.A., "Schaum's Outline of Theory and Problems of Probability and Statistics", Tata McGraw Hill Edition, 4th Edition, 2012.
5. Walpole. R.E., Myers. R.H., Myers. S.L. and Ye. K., "Probability and Statistics for Engineers and Scientists", Pearson Education, Asia, 9th Edition, 2010.
6. Kreyszig.E, "Advanced Engineering Mathematics", John Wiley and Sons,10th Edition, New Delhi, 2016.

UNIT-I [PROBABILITY & RANDOM VARIABLES]

Random Experiment: Experiments which do not produce the same result or outcome on every trial

Outcome: Results of an observation.

Trial: The performance of a random experiment is called a Trial and outcome is called event.

Sample space: The totality of the possible outcomes of a random experiment. Denoted by S

Example

① Throwing of a coin is a trial and getting H or T is an event.

$$S = \{T, H\} = \text{sample space}$$

② When Rolling a die

$$S = \{1, 2, 3, 4, 5, 6\}$$

Mutually Exclusive Events: If the occurence of one event prevents the occurence of all other events.

Example: In a experiment of throwing of dice, the occurence of any one number prevents the occurences of all other numbers.

Mutually Exclusive and Exhaustive Events:

If all the events are mutually exclusive and no other event is possible.

Example consider the experiment of tossing a coin. The possible outcomes are H and T and these two events are mutually exclusive and exhaustive.

Independent Event: If the occurence of one event has no influence over the occurence of the other.

Example: Consider the experiment of tossing two coins. The events 'getting head in the first coin' and 'getting head in the second coin' are independent.

Favourable Event: The trials which entail the happening of an event are said to be favourable to the event.

Example: In the tossing of a die, the number of favourable events to the appearance of a multiple of 3 are two (ie) 3 and 6.

Probability: Let A be the event

Probability of happening A is defined as

$$P(A) = \frac{n(A)}{n(S)}$$

where $n(A)$ = no. of elements in A

$n(S)$ = no. of elements in S

Axioms of Probability:

(1) $0 \le P(A) \le 1$ for any event A

(2) $P(S) = 1$

(3) For mutually exclusive events $E_1, E_2, E_3, \ldots$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

Important Results.

(1) Probability of an impossible event is zero

(2) $P(A^c) = 1 - P(A)$

where A^c = complementary of A

(3) If $B \subset A$ then $P(B) \le P(A)$

(4) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(or)

$P(A \cup B) = P(A) + P(B) - P(AB)$

5) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

OR)

$P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$

(6) If $A_1, A_2, \ldots, A_n$ are n mutually exclusive events then

$P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

(7) Multiplication Theorem: If two events A, B are independent and can happen simultaneously, then their joint occurance $P(A \cap B) = P(A) \cdot P(B)$.

* This thm can be extended to three or more events.

(8) If A and B are independent events, then

(i) $\bar{A}$ and $\bar{B}$ are independent

(ii) $\bar{A}$ and B are independent

(iii) A and $\bar{B}$ are independent.

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Problems

① Find the probability that exactly one head appears in a single throw of a fair coin.

Soln

$S = \{H, T\} \Rightarrow n(S) = 2$

Let A be the event of getting exactly one head in a single throw of a fair coin

$\Rightarrow A = \{H\}$

$\Rightarrow n(A) = 1.$

$\therefore P(A) = \dfrac{n(A)}{n(S)} = \dfrac{1}{2}$

② If two dice are rolled, what is the probability that the sum of the upturned faces will be equal to 7 ?

Soln

$n(S) = 36.$

Let A be the event of getting 7 when we sum the upturned faces.

$\Rightarrow A = \{(1,6), (6,1), (2,5), (5,2), (3,4), (4,3)\}.$

$\Rightarrow n(A) = 6.$

$\therefore P(A) = \dfrac{n(A)}{n(S)} = \dfrac{6}{36} = \dfrac{1}{6}.$

③ From a pack of 52 cards two cards drawn the first being replaced before the second is drawn. Find the probability that first one one is diamond and second is a king?

Soln $n(S) = 52$

Let A be the event of drawing a diamond

$\Rightarrow n(A) = 13$

$\Rightarrow P(A) = \dfrac{n(A)}{n(S)} = \dfrac{13}{52} = 1/4$

Let B be the event of drawing a king.

$\Rightarrow n(B) = 4.$

$\Rightarrow P(B) = \dfrac{n(B)}{n(S)} = \dfrac{4}{52} = \dfrac{1}{13}.$

Since A & B are independent events,

$P(A \cap B) = P(A) \times P(B)$

$= \dfrac{1}{4} \times \dfrac{1}{13}$

$= \dfrac{1}{52}$

(c) Prob. of 1st one is diamond and Second one is a king $= P(A \cap B) = 1/52$

④ A bag contains 5 white and 10 red balls. Three balls are taken out at random. Find the probability that all the three balls drawn red.

Soln

There are 15 balls in total.

$\therefore$ We can select $^{15}C_3 = \frac{15!}{3!\,12!}$ number of 3 different balls

$= \frac{13 \times \cancel{14}^{7} \times \cancel{15}^{5}}{\cancel{6}_{2}}$

$\therefore$ $n(S) = 455$

Let A be the event of getting 3 red balls.

$\Rightarrow n(A) = {}^{10}C_3 = \frac{10!}{7!\,3!} = \frac{\cancel{8}^{4} \times \cancel{9}^{3} \times 10}{\cancel{6}_{2}} = 120.$

$\therefore$ $P(A) = \frac{120}{455} = \frac{24}{91}.$

⑤ If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls, what is the probability that one of the drawn balls is white and the other two black.

Soln

$n(S) = {}^{11}C_3 = \frac{11!}{8!\,3!} = \frac{\cancel{9}^{3} \times \cancel{10}^{5} \times 11}{\cancel{6}_{2}} = 165$

Let A be the event of getting one of the drawn balls is white and, B be the event of the other two black.

$n(A) = {}^{6}C_1 = 6. \Rightarrow P(A) = 6/165$

$n(B) = {}^{5}C_2 = 10 \Rightarrow P(B) = 10/165.$

$P(A \cap B) = P(A) \times P(B) = \frac{6}{165} \times \frac{10}{165} = \frac{60}{165} = \frac{4}{11}$

(6) A lot of integrated circuit chips consists of 10 good, 4 with minor defects and 2 with major defects. Two chips are randomly chosen from a lot. What is the probability that atleast one chip is good?

Soln

Total chips = 16.

We can select 2 chips from 16 chips in $16C_2$ ways.

$$\Rightarrow n(S) = 16C_2 = \frac{16!}{14! \cdot 2!} = \frac{15 \times 16}{2} = 120.$$

$$P(\text{atleast one good}) = \left(\frac{10C_1 \times 6C_1}{120}\right) + \frac{10C_2}{120}$$

$$= \left(\frac{10 \times 6}{120}\right) + \frac{45}{120}$$

$$= \frac{105}{120}$$

$$= 7/8$$

(7) A committe of 5 persons is to be selected randomly from a group of 5 men and 10 women.

(a) Find the prob. that the committee consists of 2 men and 3 women.

(b) Find the prob. that the committee consists of all women.

Soln (a) $P(\text{2 men and 3 women}) = \frac{5C_2 \times 10C_3}{15C_5} = 0.4$

(b) $P(\text{All women}) = \frac{10C_5 \times 5C_0}{15C_5} = 0.084$

(8) Four persons are chosen at random from a group of containing 3 men, 2 women and 4 children. Show that the chance of exactly two of them will be children is $\frac{10}{21}$

Soln

Total no. of persons = 9.

$n(S) = 9C_4 = \frac{9!}{5!\,4!} = \frac{\cancel{6} \times 7 \times \cancel{8}^{2} \times 9}{\cancel{24}_{\cancel{4}}} = 126.$

Let A be the event of selecting exactly 2 children

$\therefore n(A) = 4C_2 \times 5C_2 = \frac{4!}{2!\,2!} \times 10$

$= 60.$

$\therefore P(A) = \frac{60}{126} = \frac{10}{21}$

(9) Four persons are chosen at random from a group containing 3 men, 2 women and 4 children. Find the probability that the exactly three of them will be children.?

Soln

$n(S) = 9C_4 = 126.$

Let A be the event of getting exactly 3 children.

$\Rightarrow n(A) = 4C_3 \times 5C_1 = 4 \times 5 = 20.$

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{20}{126} = \frac{10}{63}$

(10) One card is drawn from a pack of 52 cards. What is the probability that it is either a king or a queen?

Soln

$n(S) = 52$.

Let A be the event of getting a king

$\Rightarrow n(A) = 4$.

$\Rightarrow P(A) = \frac{4}{52} = \frac{1}{13}$

Let B be the event of getting a Queen.

$\Rightarrow n(B) = 4$.

$\Rightarrow P(B) = \frac{4}{52} = \frac{1}{13}$.

Since A & B are mutually exclusive events,

$P(A \cap B) = 0$.

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{13} + \frac{1}{13} - 0$

$= \frac{2}{13}$

(11) From a group of 5 first year, 4 second year and 4 third year students, 3 students are selected at random. Find the probability that they are first year or third year students?

Soln

$n(S) = 13C_3 = \frac{13!}{10!\,3!} = \frac{11 \times 12 \times 13}{6} = 286$.

A being 1^{st} year, $n(A) = 5C_3 = 10 \Rightarrow P(A) = \frac{10}{286}$.

B being 3^{rd} year, $n(B) = 4C_3 = 4 \Rightarrow P(B) = \frac{4}{286}$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{10}{286} + \frac{4}{286} - 0 = \frac{14}{286} = \frac{7}{143}$

(12) A bag contains 30 balls numbered from 1 to 30. One ball is drawn at random. Find the probability that the number of ball drawn will be a multiple of (a) 5 or 7.

(b) 3 or 7.

Soln

$n(S) = 30$.

Let A be the event of getting a ball with number which is a multiple of 5, and B be 7, C be 3.

$\Rightarrow \quad n(A) = 6, \quad n(B) = 4, \quad n(C) = 10.$

(a) $P(5 \text{ or } 7) = P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= \frac{6}{30} + \frac{4}{30} - 0$

$= \frac{1}{3}$

(b) $P(3 \text{ or } 7) = P(B \cup C)$

$= P(B) + P(C) - P(B \cap C)$

$= \frac{4}{30} + \frac{10}{30} - \frac{1}{30}$

$= \frac{13}{30}$

(13) A coin is biased so that a head is twice as likely to occur as a tail. If a coin is tossed 3 times, what is the probability of getting 2 tails and 1 head.

Soln $A \rightarrow$ event of getting 2 tails and 1 head

$P(H) = 2/3$, $P(T) = 1/3$.

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

$\Rightarrow n(S) = 8$.

$A = \{HTT, THT, TTH\}$.

$P(A) = P(HTT) + P(THT) + P(TTH)$

$= (2/3 \times 1/3 \times 1/3) + (1/3 \times 2/3 \times 1/3) + (1/3 \times 1/3 \times 2/3)$

$= \dfrac{6}{27}$

$= 2/9$.

(14) A can hit a target in 4 out of 5 shots and B can hit a target is 3 out of 4 shots. Find the probability that (i) the target being hit when both try.

(ii) the target being hit by exactly one perso

Soln

$P(A) = 4/5$, $P(B) = 3/4$.

A & B are not mutually exclusive.

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 4/5 + 3/4 - P(A).P(B)$

$= 4/5 + 3/4 - (4/5 \times 3/4)$

$= 1/5 + 3/4$

$= \dfrac{4+15}{20} = 19/20$

(ii) To find the target being hit by exactly one person.

$$= P\left[(A \cap \bar{B}) \cup (\bar{A} \cap B) \right]$$

$$= P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= \{P(A) * P(\bar{B})\} + \{P(\bar{A}) \times P(B)\}$$

$$= \left\{ \frac{4}{5} \times (1 - P(B)) \right\} + \left\{ (1 - P(A)) \times 3/4 \right\}$$

$$= \left\{ 4/5 \times (1 - 3/4) \right\} + \left\{ (1 - 4/5) \times 3/4 \right\}$$

$$= \left\{ 4/5 \times \frac{1}{4} \right\} + \left\{ \frac{1}{5} \times 3/4 \right\}$$

$$= 1/5 + \frac{3}{20}$$

$$= \frac{4+3}{20}$$

$$= 7/20.$$

(15) A total of 36 members of a club play tennis, 28 play squash and 18 play badminton. Furthermore, 22 of members play both tennis and squash, 12 play both tennis and badminton, 9 play both squash and badminton, and 4 play all the three sports. How many members of this club play atleast one of these sports?

Soln

Let N be total member.

$T \to$ tennis, $S \to$ squash, $B \to$ Badminton.

$$P(T \cup S \cup B) = P(T) + P(S) + P(B) - P(T \cap S) - P(B \cap T) - P(S \cap B) + P(T \cap S \cap B)$$

$$= \frac{36 + 28 + 18 - 22 - 12 - 9 + 4}{N}$$

$= 43/N \Rightarrow$ 43 can play atleast one sports.

(16) Event A and B are such that $P(A+B) = 3/4$, $P(AB) = 1/4$, $P(\bar{A}) = 2/3$ then find $P(B)$

Soln

$$P(A) = 1 - P(\bar{A}) = 1 - 2/3 = 1/3$$

$$P(A+B) = P(A) + P(B) - P(AB)$$

$$3/4 = 1/3 + P(B) - 1/4$$

$$\Rightarrow P(B) = 3/4 + 1/4 - 1/3 = 2/3.$$

(17) If $P(A) = 0.4$, $P(B) = 0.7$ and $P(A \cap B) = 0.3$ then find the probability that neither A nor B occurs.

Soln

$$P(\text{neither A nor B}) = P(\bar{A} \cap \bar{B})$$

$$= P(\overline{A \cup B}) \quad (\text{Using Demorgan's law})$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [0.4 + 0.7 - 0.3]$$

$$= 1 - [0.8]$$

$$= 0.2$$

(18) Prove that for any event A in S, $P(A \cap \bar{A}) = 0$.

Soln

W.K.T $P(A \cup \bar{A}) = P(S) = 1.$

$$\Rightarrow P(A) + P(\bar{A}) - P(A \cap \bar{A}) = 1.$$

$$\Rightarrow P(A) + (1 - P(A)) - P(A \cap \bar{A}) = 1 \quad (\because \text{using } P(\bar{A}) = 1 - P(A)$$

$$\Rightarrow 1 - P(A \cap \bar{A}) = 1$$

$$\Rightarrow P(A \cap \bar{A}) = 1 - 1 = 0 \qquad H/P.$$

Marginal Probability: A probability of only one event that takes place is called a marginal probability

Joint Probability: The probability of occurence of both events A and B together denoted by $P(A \cap B)$, is known as joint probability of A and B.

Conditional Probability: The conditional probability of A given B is

$$P(A/_B) = \frac{P(A \cap B)}{P(B)} \quad \text{if} \quad P(B) \neq 0$$

Note

* $$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{provided } P(A) \neq 0.$$

$$\therefore \quad P(A \cap B) = \begin{cases} P(B) \cdot P(A/B) & \text{if } P(B) \neq 0 \\ P(A) \cdot P(B/A) & \text{if } P(A) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Example: When a fair dice is tossed, the conditional probability of getting '2', given that an even number has been obtained, is equal to 1/3.

$S = \{1, 2, 3, 4, 5, 6\}$.

$A = \{2, 4, 6\}$.

$B = \{2\}$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{(1/6)}{(3/6)} = 1/3.$$

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Notes

* $\dfrac{P(A/B)}{P(B/A)} = \dfrac{P(A\cap B)/P(B)}{P(A\cap B)/P(A)} = \dfrac{P(A)}{P(B)}$

* $P(A/A) = \dfrac{P(A\cap A)}{P(A)} = \dfrac{P(A)}{P(A)} = 1$

* If A and B are independent events, then

$P(A\cap B) = P(A) \cdot P(B)$

In this case conditional probability can be defined

as $P(A/B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{P(A) \cdot P(B)}{P(B)} = P(A)$

Similarly $P(B/A) = P(B)$

* Relationship between conditional, joint and marginal probabilities.

The conditional probability of event B given that A has already happend is given by

$$P(B/A) = \frac{P(B\cap A)}{P(A)}$$

where $P(A\cap B)$ = Joint probability of events A and B happening together

Problems

① Among the workers in factory only 30% receive bonus. Among those receiving the bonus only 20% are skilled. What is the probability of a randomly selected worker who is skilled and receiving bonus?

Soln

$A = \{$The event of receiving bonus$\}$

$B = \{$The event of considering skilled workers$\}$

$$P(A) = 30\% = \frac{30}{100} = 0.3$$

$$P(B/A) = 20\% = \frac{20}{100} = 0.2$$

$$P(A \cap B) = P(A) \cdot P(B/A) = 0.3 \times 0.2 = 0.06.$$

$\therefore$ 6% of workers are skilled & receiving bonus.

② Two manufacturing plants produce similar parts. Plant I produces 1000 parts, 100 of which are defective. Plant II produces 2000 parts, 150 of which are defective. A part is selected at random and found to be defective. What is the probability that it came from plant I.

Soln

$A \rightarrow$ the part selected come from plant I

$B \rightarrow$ the part selected is defective.

$\therefore$ $A \cap B \rightarrow$ the part selected is defective and from plant I

$A/B \rightarrow$ Conditional probability of event A when the event B has already happened

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{100}{3000} = \frac{1}{30}$$

$$P(B) = \frac{n(B)}{n(S)} = \frac{250}{3000} = \frac{1}{12}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{(1/30)}{(1/12)} = \frac{12}{30} = \frac{6}{15} = \frac{2}{5}$$

(3) A bag contains 5 red 3 green balls and a second bag 4 red and 5 green balls. One of the bages is selected at random and a draw of 2 balls is made from it. What is the probability that one of them is red and the other is green?

Soln

5-R, 3-G	4-R, 5-G
Bag 1	Bag 2

Let A_1, A_2 denote the event of selecting the first bag, second bag respectively.

$$\Rightarrow P(A_1) = \frac{1}{2} = P(A_2).$$

$$\therefore S = A_1 \cup A_2.$$

Let B denote the event of selecting one red and one green ball

$$P(B/A_1) = \frac{5C_1 \times 3C_1}{8C_2} = \frac{15}{28}$$

$$P(B/A_2) = \frac{4C_1 \times 5C_1}{9C_2} = \frac{20}{\cancel{38}\ 36} = \frac{5}{\cancel{8}9}$$

Required probability $= P(A_1) \cdot P(B/A_1) + P(A_2) \cdot P(B/A_2)$

$$= \left(\frac{1}{2} \times \frac{15}{28}\right) + \left(\frac{1}{2} \times \frac{5}{9}\right)$$

$$= 275/\ldots$$

④ Find the probability of drawing two balls in succession from a bag containing 3 red and 6 black balls when

(i) the ball that is drawn first is replaced

(ii) it is not replaced.

Soln

A → event of ~~selecting~~ drawn red ball

B → event of second drawn ball is red

$P(A) = \frac{3}{9} = \frac{1}{3}$

$P(B) = 3/9 = 1/3.$

(i) If the 1st ball is replaced, then the events are independent.

$P(A \cap B) = P(A) \cdot P(B) = 1/9$

(ii) If the first ball is not replaced after taking a red ball the bag will contain only 8 balls out of which 2 are red. ∴ The events are not independent

$\Rightarrow P(A \cap B) = P(A) \cdot P(B/A)$

$= 1/3 \times (2/8) = 2/24 = 1/12$

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(5) An Urn contains 10 white and 3 black balls. Another urn contains 3 white and 5 black balls. Two balls are drawn at random from the first urn and placed in the second urn and then 1 ball is taken at random from the latter. What is the probability that it is a white ball?

Soln

Let B_1 be event of drawing 2 white balls from 1st urn

B_2 be event of drawing 2 black balls from 1st urn

B_3 be event of drawing 1 white and 1 black ball from 1st urn

$\therefore$ B_1, B_2, B_3 are mutually exclusive and exhaustive

Let A be an event of drawing a white ball from the 2nd urn after transfer.

$$P(B_1) = \frac{10C_2}{13C_2} = \frac{15}{26}.$$

$$P(B_2) = \frac{3C_2}{13C_2} = \frac{1}{26}$$

$$P(B_3) = \frac{10C_1 \times 3C_1}{13C_2} = \frac{10 \times 3}{(12 \times 13/2)} = \frac{\cancel{2} \times 10 \times \cancel{3}}{\cancel{12} \times 13} = \frac{10}{26} = 5/13.$$

$$P(A/B_1) = \frac{5}{10}, \quad P(A/B_2) = 3/10, \quad P(A/B_3) = 4/10.$$

$$\therefore P(A) = P(B_1) \times P(A/B_1) + P(B_2) \times P(A/B_2) + P(B_3) \times P(A/B_3)$$

$$= \left(\frac{15}{26} \times \frac{5}{10}\right) + \left(\frac{1}{26} \times \frac{3}{10}\right) + \left(\frac{5}{13} \times 4/10\right)$$

$$= \frac{75}{260} + \frac{3}{260} + \frac{20}{130}$$

$$= \frac{78 + 40}{260}$$

$$= 118/260 = 59/130.$$

Bayes's Theorem

Let $B_1, B_2, \ldots B_n$ be an exhaustive and mutually exclusive random experiments and A be an event related to that B_i then

$$P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)}$$

Proof:

We know that $P(A \cap B_i) = P(B_i) \times P(A/B_i)$

Also $P(A \cap B_i) = P(A) \times P(B_i/A)$

$$\therefore \quad P(A) \times P(B_i/A) = P(B_i) \times P(A/B_i) \longrightarrow ①$$

$$P(B_i/A) = \frac{P(B_i \cap A)}{P(A)}$$

$$= \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}$$

H/P.

~~ood~~

Note:

Use Bayes's thm for the case of finding the probability ~~to find~~ $P(B_i/A)$ when B_i already happened.

Problems

① The contents of urns I, II, III are as follows

	White	Black	Red
Urn I	1	2	3
Urn II	2	1	1
Urn III	4	5	3

One urn is selected at random and two balls are drawn. They happen to be white and red. What is the probability that they come from urns I, II and III ?

Soln

$B_1 \rightarrow$ event of selected urn is I

$B_2 \rightarrow$ event of " " " II

$B_3 \rightarrow$ event of " " " III

$\Rightarrow$ $P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$.

$A \rightarrow$ event of two balls selected are white and red.

$$P(A/B_1) = \frac{1C_1 \times 3C_1}{6C_2} = \frac{3}{15} = \frac{1}{5}$$

$$P(A/B_2) = \frac{2C_1 \times 1C_1}{4C_2} = \frac{2}{6} = \frac{1}{3}$$

$$P(A/B_3) = \frac{4C_1 \times 3C_1}{12C_2} = \frac{12}{66} = \frac{2}{11}$$

Using Baye's thm, $P(B_i/A) = \dfrac{P(B_i) \cdot P(A/B_i)}{\sum_{i=1}^{3} P(B_i) \cdot P(A/B_i)}$

Put $i=1$

Prob of selected 2 balls (1 white & 1 red) came from

Urn I $= P(B_1/A) = \dfrac{P(B_1) \times P(A/B_1)}{\sum_{i=1}^{3} P(B_i) \times P(A/B_i)}$

$$= \frac{\left(\frac{1}{3} \times \frac{1}{5}\right)}{\left(\frac{1}{3} \times \frac{1}{5}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{2}{11}\right)} = \frac{33}{118}$$

Probability of selected balls (1 white & 1 red) came from Urn II $= P(B_2/A) = \dfrac{P(B_2) \times P(A/B_2)}{\sum_{i=1}^{3} P(B_i) \times P(A/B_i)}$

$$= \frac{\left(\frac{1}{3} \times \frac{1}{3}\right)}{\left(\frac{1}{3} \times \frac{1}{5}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{2}{11}\right)} = \frac{55}{118}$$

Probability of selected balls (1 white & 1 red) came from Urn III $= P(B_3/A) = \dfrac{P(B_3) \times P(A/B_3)}{\sum_{i=1}^{3} (P(B_i)) \times P(A/B_i)}$

$$= \frac{\left(\frac{1}{3} \times \frac{2}{11}\right)}{\left(\frac{1}{3} \times \frac{1}{5}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{2}{11}\right)} = \frac{30}{118}.$$

<u>Aliter</u> to find $P(B_3/A)$

W.K.T Total probability $= 1$.

$$P(B_1/A) + P(B_2/A) + P(B_3/A) = 1.$$

$$\Rightarrow \frac{33}{118} + \frac{55}{118} + P(B_3/A) = 1$$

$$\Rightarrow P(B_3/A) = 1 - \frac{33}{118} - \frac{55}{118}$$

$$= \frac{118 - 33 - 55}{118}$$

$$= \frac{30}{118}$$

(2) In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of total output. Also out of these output of A, B, C, 5, 4, 2 percent respectively are defective. A bolt is drawn at random from a total output and it is found to be defective. What is the probability that it was manufactured by the machine B?

Soln

Let E_1, E_2, E_3 be the events that the bolts are manufactured by A, B, C respectively.

$P(E_1) = 25\% = \frac{25}{100} = 0.25$

$P(E_2) = 35\% = \frac{35}{100} = 0.35$

$P(E_3) = 40\% = \frac{40}{100} = 0.4$

Let A be the event of drawing defective bolt.

Given $P(X/E_1) = \frac{5}{100} = 0.05$

$P(X/E_2) = \frac{4}{100} = 0.04$

$P(X/E_3) = 2/100 = 0.02$

To find:

The probability that the defective bolt selected at random is manufactured ~~by~~ from machine B $= P(E_2/X)$

By Baye's thm

$$P(E_2/X) = \frac{P(E_2) \cdot P(X/E_2)}{\sum_{i=1}^{3} P(E_i)\, P(X/E_i)}$$

$$= \frac{(0.35)(0.04)}{(0.25 \times 0.05) + (0.35 \times 0.04) + (0.4 \times 0.02)}$$

$$= 0.406$$

(3) A certain firm has plant A, B and C producing IC chips. Plant A produces twice the output from B, and B produces twice the output from C. The probability of a non-defective product produced by A, B, C are respectively 0.85, 0.75, and 0.95. A customer receives a defective product. Find the probability that it came from plant B?

Soln

Let plant A produces 100 number of IC chips

Let plant B " 50 , ...

Let plant C " 25 . . . .

$\Rightarrow$ $P(A) = \frac{100}{175} = 0.57$

$P(B) = \frac{50}{175} = 0.29$

$P(C) = \frac{25}{175} = 0.14$

Let E be the event that the item produced is non-defective

$\therefore$ $\bar{E}$ denotes the event that the item produced is defective.

Given $P(E/A) = 0.85 \Rightarrow P(\bar{E}/A) = 1-0.85 = 0.15$

$P(E/B) = 0.75 \Rightarrow P(\bar{E}/B) = 1-0.75 = 0.25$

$P(E/C) = 0.95 \Rightarrow P(\bar{E}/C) = 1-0.95 = 0.05$

Prob. of customer receives a defective product from plant B

$$= P(B/\bar{E}) = \frac{P(B) \cdot P(\bar{E}/B)}{P(A) \cdot P(\bar{E}/A) + P(B) \cdot P(\bar{E}/B) + P(C) \cdot P(\bar{E}/C)}$$

$$= \frac{(0.29) \times (0.25)}{[(0.57 \times 0.15) + (0.29 \times 0.25) + (0.14 \times 0.05)]} = 0.4394$$

(4) A box contains 7 red and 13 blue balls. Two balls are selected at random and are discarded without their colours being seen. If a third ball is drawn randomly and observed to be red, what is the probability that both of the discarded balls were blue?

Soln

Let BB = event the discarded balls are Blue, Blue

BR = - - - - - - - - - Blue, Red

RR = - - - - - - - - - Red, Red

R = event that 3rd ball drawn is Red.

$$P(BB) = \frac{13C_2}{20C_2} = \frac{78}{190} = \frac{39}{95}$$

$$P(BR) = \frac{7C_1 \times 13C_1}{20C_2} = \frac{7\times13}{190} = \frac{91}{190}$$

$$P(RR) = \frac{7C_2}{20C_2} = \frac{21}{190}.$$

Using Baye's thm

$$P(BB/R) = \frac{P(R/BB) \times P(BB)}{P(R/BB)\times P(BB) + P(RR)\,P(R/RR) + P(R/BR) \times P(BR)}$$

$$= 0.46$$

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Random Variable: It is a function from sample space to R.

It is denoted by X

Thus $X : S \to R$

~~Also it denoes~~

Discrete Random Variable: If X takes atmost countable number of values in R then X is discrete random variable.

Continuous Random Variable: If X takes all possible values b/w certain limits say from real number 'a' and 'b' then it is called continuous random Variable.

Probability Mass Function (P.m.f)

If X is a discrete random variable, then the function $P(x) = P(X = x)$ is called the probability mass function of X.

Probability Distribution

The values assumed by the random Variable X presented with corresponding probabilities is known as the probability distribution of X

X	x_1	x_2	x_3	...
$P(X=x)$	P_1	P_2	P_3	...

Cumulant Distribution (or) Distribution Function of X

The cumulant destribution function $F(x)$ of a discrete random variable X with probability distribution $P(x)$ is given by

$$F(x) = P(X \leq x) = \sum_{t \leq x} p(t)$$

where $x = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots \infty$

Important Results

① $P(x_1 < X \leq x_2) = F(x_2) - F(x_1)$

② $P(X \leq \infty) = 1$

③ $P(X \leq -\infty) = 0$

④ If $x_1 \leq x_2$ then $P(X = x_1) \leq P(X = x_2)$

⑤ $P(X > x) = 1 - P(X \leq x)$

⑥ $P(X \leq x) = 1 - P(X > x)$

Expected Value of a discrete random variable X

Let X be a discrete random variable assuming values $x_1, x_2, x_3, \ldots x_n$ with corresponding probabilities $P_1, P_2 \ldots P_n$

$$E(x) = \sum_i x_i P_i$$ is called expected value of X.

$E(x) \rightarrow$ mean of a r.v X

Variance of a R V

$$Var(x) = E(x^2) - [E(x)]^2$$

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Important Formulae

1) Mean $= E(x) = \sum_i x_i P(x_i)$ = Expected Value

2) $E(x^2) = \sum_i x_i^2 P(x_i)$

3) Variance $= Var(X) = E(x^2) - [E(x)]^2$

4) $E(aX+b) = aE(X)+b$

5) $Var(ax \pm b) = a^2 Var(x)$

6) Probability mass function $p(x) = P(x=x)$

7) Standard deviation $= \sqrt{Var(X)}$

Problems

① For the following probability distribution

(i) Find the distribution function of X

(ii) What is the smallest value of x for which $P(x \le x) > 0.5$

$X=x_i$	0	1	2
$P(x=x_i)$	1/4	2/4	1/4

Soln

(i)

x_i	$P(x_i)$	$F(x_i) = P(X \le x_i)$
0	1/4	$F(0) = 1/4$
1	2/4	$F(1) = 1/4 + 2/4 = 3/4$
2	1/4	$F(2) = 1/4 + 2/4 + 1/4 = 1$.

(ii) The smallest value of x for which $P(X \le x) > 0.5$

is 1.

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(2) The number of hardware failures of a computer system in a week of operations has the following pmf

Number of Failures	0	1	2	3	4	5	6
Probability	.18	.28	.25	.18	.06	.04	.01

Find the mean number of failures in a week

Soln

$$\text{Mean} = E(X) = \sum x_i P(x_i)$$

$$= (0\times0.18) + (1\times0.28) + (2\times .25) + (3\times0.18) + (4\times0.06)$$

$$+ (5\times0.04) + (6\times0.01)$$

$$= 1.82$$

(3) Determine the mean, Variance, $E(2X+1)$, $Var(2X+1)$ of a discrete random Variable X given its distribution as follows.

$X = x_i$	1	2	3	4	5	6
$F(X=x_i)$	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	1

x_i	$F(x)$ $(=P(X\le x))$	$P(x_i)$	$x_i P(x_i)$	x_i^2	$x_i^2 P(x_i)$
1	$F(1) = 1/6$	$P(1) = F(1) = 1/6$.	$1/6$	1	$1/6$
2	$F(2) = 2/6$	$P(2) = F(2) - F(1) = 1/6$.	$2/6$	4	$4/6$
3	$F(3) = 3/6$	$P(3) = F(3) - F(2) = 1/6$	$3/6$	9	$9/6$
4	$F(4) = 4/6$	$P(4) = F(4) - F(3) = 1/6$	$4/6$	16	$16/6$
5	$F(5) = 5/6$	$P(5) = F(5) - F(4) = 1/6$	$5/6$	25	$25/6$
6	$F(6) = 1$	$P(6) = F(6) - F(5) = 1/6$	$6/6$	36	$36/6$
			$\Sigma x_i P(x_i) = 21/6$		$\Sigma x_i^2 P(x_i) = 91/6$

(i) $E(x) = \text{Mean} = \sum_i x_i P(x_i)$

$= 21/6 = 7/2$

(ii) $E(x^2) = \sum_i x_i^2 P(x_i) = 91/6$

(iii) $Var(x) = \Sigma(x^2) - [E(x)]^2$

$= \frac{91}{6} - \frac{441}{36}$

$= \frac{546-441}{36}$

$= \frac{105}{36}$

$= \frac{35}{12}$

(iv) $E(2x+1) = 2E(x) + 1$

$= 2(7/2) + 1 = 8$

$\cancel{= 14/3 + 1}$

$\cancel{= 17/3}$

(v) $Var(2x+1) = 2^2 Var(x) = 4\, Var(x) = \cancel{4} \times \frac{35}{\cancel{12}\,3} = \frac{35}{3}$

④ Determine the constant K given the following probability distribution of discrete random Variable X. Also find mean and variance of X.

$X = x_i$	1	2	3	4	5	Total
$P(X=x_i)$	0.1	0.2	K	2K	0.1	1.0

Soln W.K.T $\sum_{x_i} P(x_i) = 1$

$\Rightarrow 0.1 + 0.2 + K + 2K + 0.1 = 1$

$\Rightarrow 3K = 1 - 0.4 = 0.6 \Rightarrow \boxed{K = 0.2}$

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Mean $= E(X) = \sum x_i P(x_i) = (1 \times 0.1) + (2 \times 0.2) + 3k + 8k + 0.5$

$= 0.1 + 0.4 + 0.6 + 1.6 + 0.5 \quad (\because k = 0.2)$

$= 3.2$

$E(X^2) = \sum_i x_i^2 P(x_i) = (1^2 \times 0.1) + (2^2 \times 0.2) + (3^2 \times 0.2) + (4^2 \times 0.2)$

$+ (5^2 \times 0.1)$

$= 11.6$

$Var(X) = E(X^2) - [E(X)]^2$

$= 11.6 - (3.2)^2$

$= 1.36$

5 A random Variable X has the following distribution.

$X = x_i$	-2	-1	0	1	2	3
$P(X=x_i)$	0.1	k	0.2	$2k$	0.3	$3k$

Find (i) The value of k

(ii) Evaluate $P(X<2)$ and $P(-2<X<2)$

(iii) Find the cumulative distribution of X

(iv) Evaluate the mean of X

Soln.

(i) W.K.T $\sum_i P(x_i) = 1$

$\Rightarrow 0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$

$\Rightarrow 6k + 0.6 = 1$

$\Rightarrow 6k = 0.4$

$\Rightarrow k = \frac{0.4}{6} = 1/15$

(ii) $P(X<2) = P(X=-2) + P(X=-1) + P(X=0) + P(X=1)$

$= 0.1 + k + 0.2 + 2k$

$= 0.1 + 1/15 + 0.2 + 2/15 \quad (\because k = 1/15)$

$= 1/2$

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To find $P(-2 < X < 2)$

$$P(-2 < X < 2) = P(X=-1) + P(X=0) + P(X=1)$$
$$= K + 0.2 + 2k$$
$$= 1/15 + 0.2 + 2/15 \quad (\because k = 1/15)$$
$$= 2/5$$

(iii) To find the cumulative distribution of X

x_i	$P(x_i) = P(X=x_i)$	$F(x_i) = P(X \le x_i)$
-2	0.1	$F(-2) = P(X \le -2) = 0.1$
-1	$1/15$	$F(-1) = P(X \le -1) = 0.1 + 1/15 = 0.17$
0	0.2	$F(0) = P(X \le 0) = 0.17 + 0.2 = 0.37$
1	$2/15$	$F(1) = P(X \le 1) = 0.37 + 2/15 = 0.5$
2	0.3	$F(2) = P(X \le 2) = 0.5 + 0.3 = 0.8$
3	$3/15$	$F(3) = P(X \le 3) = 0.8 + 3/15 = 1$

(iv) Mean of $X = \sum_i x_i P(x_i)$

$$= (-2 \times 0.1) + (-1 \times 1/15) + (0 \times 0.2) + (1 \times 2/15)$$
$$+ (2 \times 0.3) + (3 \times 3/15)$$
$$= 16/15$$

⑥ A random variable x has the following probability function

$X = x_i$	0	1	2	3	4	5	6	7
$P(X=x_i)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find (i) The value of k

(ii) Evaluate $P[X<6]$, $P[X \geq 6]$

(iii) If $P[X \leq c] = 1/2$, then find the minimum value of c

(iv) Evaluate $P[1.5 < X < 4.5 / X > 2]$

(v) Find $P[X<2]$, $P[X>3]$, $P[1<X<5]$

Soln.

(i) W.K.T $\sum_i P(x_i) = 1$

$\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow 9k + 10k^2 = 1$

$\Rightarrow 9k + 10k^2 - 1 = 0$

10	-1
$10k$	$10k$

$\Rightarrow 10k^2 + 9k - 1 = 0$

$\Rightarrow (10k + 10)(10k - 1) = 0$

$\Rightarrow (k = -1)$ or $(k = 1/10)$

Since probability cannon be $-$ve, we ignore $k = -1$

$\therefore \boxed{k = 1/10}$

(ii) $P[X<6] = \sum_{i=1}^{5} P(X = x_i) = 0 + 1/10 + 2/10 + 2/10 + 3/10 + 1/100$

$= 8/10 + 1/100$

$= 81/100.$

$P(X \geq 6) = 1 - P(X<6) = 1 - 81/100 = 19/100.$

(iii) First we find the cumulant distribution function of X

x_i	$P(x_i) = P(X = x_i)$	$F(x_i) = P(X \le x_i)$
0	0	$F(0) = P(X \le 0) = 0$
1	$1/10$	$F(1) = P(X \le 1) = 0 + 1/10 = 1/10$
2	$2/10$	$F(2) = P(X \le 2) = 1/10 + 2/10 = 3/10$
3	$2/10$	$F(3) = P(X \le 3) = 3/10 + 2/10 = 5/10 \; (= 1/2)$
4	$3/10$	$F(4) = P(X \le 4) = 5/10 + 3/10 = 8/10 > 1/2$
5	$1/100$	$F(5) = P(X \le 5) = 8/10 + 1/100 = 81/100 > 1/2$
6	$2/100$	$F(6) = P(X \le 6) = 81/100 + 2/100 = \frac{83}{100} > 1/2$
7	$17/100$	$F(7) = P(X \le 7) = \frac{83}{100} + \frac{17}{100} = 1 > 1/2$.

From the above table we can calculate minimum C so that $P(X \le C) > \frac{1}{2}$

Minimum Value of $C = 4$

(iv) $P\left[1.5 < X < 4.5 / X > 2\right] = \dfrac{P\left[(1.5 < X < 4.5) \cap (X > 2)\right]}{P(X > 2)}$

$$= \frac{P(3) + P(4)}{1 - P(X \le 2)} = \frac{\left(2/10 + 3/10\right)}{\left(1 - \left\{0 + 1/10 + 2/10\right\}\right)} = 5/7.$$

(b) (i) $P(X<2) = P(X=0) + P(X=1)$ $\cancel{+P(x=2)}$

$= 0 + k$

$= 0 + 1/10$

$= 1/10$.

$P(X>3) = 1 - P(X \le 3)$

$= 1 - \left[P(X=0) + P(X=1) + P(X=2) + P(X=3)\right]$

$= 1 - [0 + k + 2k + 2k]$

$= 1 - 5k$

$= 1 - 5 \times 1/10$

$= 1/2$.

$P(1<X<5) = P(X=2) + P(X=3) + P(X=4)$ $\cancel{+P}$

$= 2k + 2k + 3k$

$= 7k$

$= 7/10$.

7) If $Var(X) = 4$, then find $Var(3X+8)$, where X is a random variable.

Soln

$Var(aX+b) = a^2 Var(X)$

$\therefore Var(3X+8) = 3^2 Var(X)$

$= 9 \times Var(X)$

$= 9 \times 4$ $(\because Var(X) = 4)$

$= 36$.

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Probability Density Function: (P.d.f)

For a continuous random variable X, a probability density function is a function such that

(1) $f(x) \geq 0$ (2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx$ = area under $f(x)$ from a to b where a & b are real numbers.

Cumulative distribution function: The cumulative distribution function of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx \quad \text{for } -\infty < x < \infty$$

Note

* The probability density function of a continuous random variable can be determined from the cumulative distribution function by differentiating.

$$\frac{d}{dx}\int_{-\infty}^{x} f(t)\,dt = f(x).$$

(OR)

$$\frac{d}{dx}\left[F(x)\right] = f(x).$$

Mean (or) Expected Value of a random variable (X)

$$E(X) = \text{mean} = \int_{-\infty}^{\infty} x f(x)\,dx$$

Similarly $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$Var(x) = E(x^2) - [E(x)]^2$

Standard deviation $= \sigma = \sqrt{Var(x)}$

Important Formula

1) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

2) $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\, dt$

3) $P(a \le X \le b) = F(b) - F(a)$

4) $P(a \le X \le b) = P(a < X \le b) = P(a \le X < b) = P(a < X < b)$

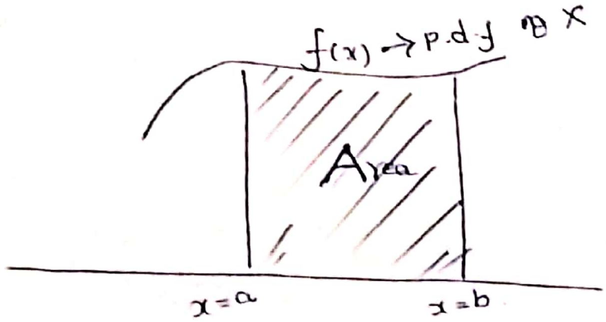
for a continuous random variable.

5) $0 \le F(x) \le 1$

6) $F(x)$ is a non-decreasing function of X

7) $F(-\infty) = 0$

8) $F(\infty) = 1$

9) 

$$\text{Area} = \int_a^b f(x)\, dx = P(a \le X \le b)$$

Problems

① A continuous random variable X has p.d.f $f(x) = k, 0 < x < 1$

Determine the constant k. Find $P\left[X \le \frac{1}{4}\right]$

Soln

$$f(x) = k \; ; \; 0 < x < 1$$

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_0^1 f(x)\,dx = 1 \quad \Rightarrow \int_0^1 k\,dx = 1$$

$$\Rightarrow \boxed{k = 1}$$

$$P\left[X \le \frac{1}{4}\right] = \int_{-\infty}^{1/4} f(x)\,dx = \int_0^{1/4} 1\,dx = \frac{1}{4}$$

② If $f(x) = \begin{cases} kxe^{-x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$ is a p.d.f of a random variable then find k.

Soln

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_0^{\infty} kxe^{-x} = 1 \quad \Rightarrow k\left\{x\left(\frac{e^{-x}}{-1}\right) - (1)\left(\frac{e^{-x}}{1}\right)\right\}_{x=0}^{x=\infty} = 1$$

$$\Rightarrow k\left\{[0-0] - [0-1]\right\} = 1$$

$$\Rightarrow k = 1$$

Problems

(1) A continuous random variable X has pdf $f(x)=k, 0 \le x \le 1$

Determine the constant k. Find $P[X \le \frac{1}{4}]$

Soln

$$f(x) = k \; ; \; 0 \le x \le 1$$

WKT $$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \int_{0}^{1} f(x).\, dx = 1 \quad \Rightarrow \int_{0}^{1} k\, dx = 1$$

$$\Rightarrow \boxed{k = 1}$$

$$P\left[X \le \tfrac{1}{4}\right] = \int_{-\infty}^{1/4} f(x).\, dx = \int_{0}^{1/4} 1\, dx = \tfrac{1}{4}.$$

(2) If $f(x) = \begin{cases} kxe^{-x}, & x>0 \\ 0 & , \text{elsewhere} \end{cases}$ is a p.d.f of a random

variable then find k.

Soln

WKT $$\int_{-\infty}^{\infty} f(x).\, dx = 1$$

$$\Rightarrow \int_{0}^{\infty} kxe^{-x} = 1 \quad \Rightarrow k\left\{x.\left(\frac{e^{-x}}{-1}\right) - (1)\left(\frac{e^{-x}}{1}\right)\right\}_{x=0}^{x=\infty} = 1$$

$$\Rightarrow k\left\{[0-0]-[0-1]\right\} = 1$$

$$\Rightarrow k = 1$$

Soln :-

(i) WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} k(x - x^2)\,dx = 1$

$$\Rightarrow k\int_0^1 (x - x^2)\,dx = 1$$

$$\Rightarrow k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$\Rightarrow k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \Rightarrow k\left[\frac{1}{6}\right] = 1$$

$$\Rightarrow \boxed{k = 6}$$

(ii) To find the cumulative distribution function $(F(x))$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^x 6(x - x^2)\,dx = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)$$

$$= 3x^2 - 2x^3 \text{ for } 0 \le x \le 1$$

(iii) To find the value of a such that $P(X > a) = P(X < a)$

$$P(X > a) = \int_a^1 6(x - x^2)\,dx$$

$$P(X < a) = \int_0^a 6(x - x^2)\,dx$$

$$\therefore 6\int_0^a (x - x^2)\,dx = 6\int_a^1 (x - x^2)\,dx$$

$$6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^a = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_a^1$$

$$\cancel{6}\left[\frac{a^2}{2} - \frac{a^3}{3}\right] = \cancel{6}\left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{a^2}{2} - \frac{a^3}{3}\right)\right]$$

$$2\left[\frac{a^2}{2} - \frac{a^3}{3}\right] = \frac{1}{6}$$

$\Rightarrow \quad 3a^2 - 2a^3 = 1/2$

$\Rightarrow \quad 6a^2 - 4a^3 - 1 = 0$

Solving $\boxed{a = 1/2}$

(iv) To find the probability $P\left[X \leq 1/2 \,/\, 1/3 < X < 2/3\right]$.

Let A be event of $X \leq 1/2$,

B be event of $1/3 < X < 2/3$.

$\therefore$ ~~P(A) = P(X ≤ 2) = ∫~~

$$P(A/B) = \frac{P(A \cap B)}{P(B)}.$$

$$\therefore P(B) = \int_{1/3}^{2/3} 6(x - x^2)\,dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{2/3} = \frac{13}{54}$$

$$P(A \cap B) = \int_{1/3}^{1/2} 6(x - x^2)\,dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{1/2} = \frac{13}{27}$$

$$P(A/B) = \frac{(13/27)}{(13/54)} = 1/2$$

Binomial Distribution

* This distribution was discovered by James Bernoulli

* Each trial has two possible outcomes, generally called success and failure. Such trail is known as Bernoulli trial

  Sample space for Bernoulli trial is $S = \{s, f\}$

Example

1. A toss of a single coin (head or tail)
2. The throw of a die (even or odd number)

* An experiment consisting of a repeated number of Bernoulli trials is called Binomial experiment.

  A binomial experiment must possess the following properties

  (i) There must be fixed number of trials

  (ii) All trials must have identical probabilities of success (p)

  (iii) The trials must be independent of each other

* Consider a set of n independent Bernoulli trials in which the probability p of success in any trial is constant for each trial. Then $q = 1-p$ is the probability of failure in any trial.

* A random variable X is said to follow binomial distribution if it assumes only non-ve values and its probability mass function is given by

$$P(X=x) = \begin{cases} nC_x\, p^x q^{n-x} & , x = 0,1,2,\ldots,n, \quad q = 1-p \\ 0 & ; \text{otherwise.} \end{cases}$$

* The two independent constants n and p are known as parameters of Binomial distribution.

* If this experiment is repeated N times, the frequency function of binomial distribution is given by

$$f(x) = N\,p(x) = N \cdot {}^{n}C_{x}\, p^{x} q^{n-x}, \quad x = 0, 1, 2, 3 \dots n.$$

Important Notes

* Each trial results in two mutually disjoint outcomes, termed as success and failure
* The number of trials is finite
* The trials are independent of each other
* The probability of success (p) is constant for each trial.

The above are the experimental conditions to get the Binomial distribution.

Mode of binomial distribution

(i) If $(n+1)p$ is not an integer then

mode = integral part of $(n+1)p$.

(ii) If $(n+1)p$ is an integer

then $(n+1)p$ & $(n+1)p - 1$ are two modes.

Derivation of Mean & Variance (For Binomial distn)

$$P(X=x) = {}^nC_x\, p^x q^{n-x}$$

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx}\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x\, (pe^t)^x\, q^{n-x}$$

$$M_x(t) = (pe^t+q)^n \longrightarrow ①$$

Diff ① w.r.t 't'

$$\frac{d}{dt}\left[M_x(t)\right] = n\,(pe^t+q)^{n-1}\,pe^t$$

$$= np\left[(pe^t+q)^{n-1} \times e^t\right]$$

$$\therefore \frac{d^2}{dt}\left[M_x(t)\right] = np\left[(n-1)\,(pe^t+q)^{n-2}\,pe^{2t} + (pe^t+q)^{n-1}\,e^t\right]$$

Now mean $= E(X) = \left[\frac{d}{dt}\left(M_x(t)\right)\right]_{\text{at } t=0}$

$$= np\,(p+q)^{n-1}$$

$$\boxed{\text{Mean} = np}$$

$$E\left[x^2\right] = \left[\frac{d^2}{dt^2}\left(M_x(t)\right)\right] = np\left[(n-1)\,(p+q)^{n-2}\,p + (p+q)^{n-1}\right]$$

$$= np\left[(n-1)p + 1\right]$$

$$= (np)^2 - np^2 + np$$

$$= (np)^2 + np\,(1-p)$$

$$= (np)^2 + npq$$

$$\text{Var}(x) = E(x^2) - \left[E(x)\right]^2 = (np)^2 + npq - (np)^2$$

$$\boxed{\text{Var} = npq}$$

Problems

① A machine manufacturing screws is known to produce 5% defective. In a random sample of 15 screws, what is the probability that there are

(i) exactly 3 defectives

(ii) not more than 3 defectives.

Soln

This follow binomial distribution with

$n = 15$, and p = propability of defective $= 5\% = 0.05$

$\Rightarrow q = 1 - p = 0.95$

Let X denote the number of defectives.

(i) To find exactly 3 defectives

$$P(X=3) = 15C_3 (0.05)^3 \cdot (0.95)^{12}$$

$$= 0.0307$$

(ii) To find not more than 3 defectives $= P(X \leq 3)$

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 15C_0 (0.05)^0 (0.95)^{15} + 15C_1 (0.05)^1 (0.95)^{14}$$

$$+ 15C_2 (0.05)^2 (0.95)^{13} + (15C_3) (0.05)^3 (0.95)^{12}$$

$$= 0.994$$

Poisson Distribution

* It was introduced by S.D. Poisson

* $P(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$, $x = 0, 1, 2 \ldots n$

* $\lambda \to$ parameter of poisson distribution.

  $\lambda = np$

  $n \to$ number of independent trials

  $p \to$ probability of success (very low)

* $N \cdot \dfrac{e^{-\lambda}\lambda^{x}}{x!}$ is corresponding frequency distribution

* The following are some of the examples where the Poisson probability law can be applied.

  1) Number of defective items produced in the factory
  2) Number of deaths due to a rare disease
  3) Number of deaths due to kick of a horse in an army
  4) Number of mistakes committed by a typist per page

* If n is small we use Binomial distribution

  If n is large we use Poisson distribution

Derivation of Mean and Variance (For Poisson distn)

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$M_x(t) = e^{-\lambda} e^{\lambda e^t}$$

$$M_x(t) = e^{\lambda(e^t - 1)}$$

$$\frac{d}{dt}\left[M_x(t)\right] = e^{-\lambda}\left[e^{\lambda e^t} \lambda e^t\right]$$

Put $t=0$, $\left\{\frac{d}{dt}\left[M_x(t)\right]\right\}_{t=0} = e^{-\lambda}\left(\lambda e^{\lambda}\right) = \lambda$

$\Rightarrow$ mean $= E(x) = \frac{d}{dt}\left\{M_x(t)\right\}_{\text{at } t=0} = \lambda$

Similarly $\frac{d^2}{dt^2}\left(M_x(t)\right) = e^{-\lambda}\left[e^{\lambda e^t}(\lambda e^t)^2 + e^{\lambda e^t} \lambda e^t\right]$

$$\left\{\frac{d^2}{dt^2}\left(M_x(t)\right)\right\}_{t=0} = e^{-\lambda}\left[e^{\lambda} . \lambda^2 + e^{\lambda} \lambda\right]$$

$$E(x^2) = \lambda^2 + \lambda$$

$$Var(x) = E(x^2) - \left[E(x)\right]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{Var(x) = \lambda}$$

Scanned with OKEN Scanner

Problems

① The number of monthly breakdown of a computer is a random variable having a Poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month

(1) without a breakdown

(2) with only one breakdown

(3) with atleast one breakdown.

Soln

Given mean $= \lambda = 1.8$

Let X denotes the no of breakdown of a computer in a month

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-1.8}(1.8)^{x}}{x!}$$

(a) P (without a breakdown) $= P(X=0) = \frac{e^{-1.8}(1.8)^{0}}{0!}$

$= 0.1653$

(b) P (with only one breakdown) $= P(X=1) = \frac{e^{-1.8}(1.8)^{1}}{1!}$

$= 0.2975$

(c) P (with atleast one breakdown) $= P(X \geq 1)$

$= 1 - P(X<1)$

$= 1 - P(X=0)$

$= 1 - 0.1653$

$= 0.8347$

② Messages arrive at a switch board in a Poisson manner at an average of six per hour. Find the probability for each of the following events :

(1) exactly two messages arrive within one hour

(2) No message arrives within one hour

(3) atleast three messages arrive within one hour

Soln

Given λ = mean = 6

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-6}6^{x}}{x!}$$

(1) P(exactly two messages) $= P(X=2) = \frac{e^{-6}6^{2}}{2!} = 0.0446$

(2) P(no message arrives) $= P(X=0) = \frac{e^{-6}6^{0}}{0!} = 0.0025$

(3) P(atleast three messages) $= P(X \geq 3)$

$= 1 - P(X<3)$

$= 1 - \{P(X=0) + P(X=1) + P(X=2)\}$

$$= 1 - \left\{\frac{e^{-6}6^{0}}{0!} + \frac{e^{-6}6^{1}}{1!} + \frac{e^{-6}6^{2}}{2!}\right\}$$

$= 1 - e^{-6}\{1 + 6 + 18\}$

$= 1 - e^{-6}(25)$

$= 0.9380$

Normal Distribution

* It was introduced by French mathematician Abraham De Moivre

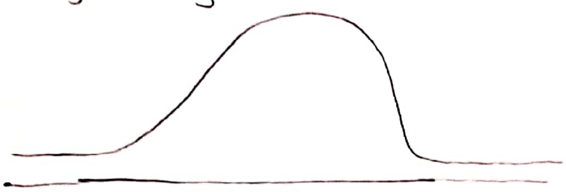
* It is a continuous distribution

* $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is it pdf

  mean $= \mu$, Variance $= \sigma^2$, Standard deviation $= \sigma$

* III The area under normal curve is unity.

* Diagram of normal curve



(Bell shaped)

* The normal distn is symmetrical distn.

* The curve has a single peak point (i.e) the distn is unimodal)

* The mean of the normal distn lies at the centre of normal curve.

* mean = median = mode for normal distn.

* The tails of the normal distn extend indefinitely and never touch the horizontal axes

* Its parameters are μ & σ.

* 67 % of observations will lies b/w $\mu \pm \sigma$
  95 % of observations will lie b/w $\mu \pm 2\sigma$
  99 % of observations will lie b/w $\mu \pm 3\sigma$

  It is known as **Area property**

* If X is normally distributed random variable μ and σ are respectively its mean and standard deviation, then $Z = \frac{X-\mu}{\sigma}$ is called standard normal random variable with mean $=0$, variance $=1$

* If $X_1, X_2, \ldots, X_n$ are independent normal variates with parameters $(m_1, \sigma_1), (m_2, \sigma_2), \ldots, (m_n, \sigma_n)$ then $X_1 + X_2 + \cdots + X_n$ is also normal variate with parameter $\left(\sum_{i=1}^{n} m_i, \sqrt{\sum_{i=1}^{n} \sigma_i^2}\right)$

Problems

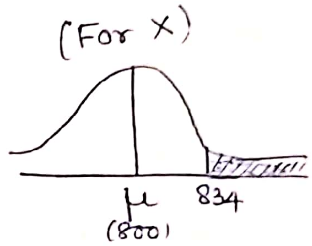
① An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hrs and a standard deviation of 40 hrs. Find

(1) the probability that a bulb burns more than 834 hours

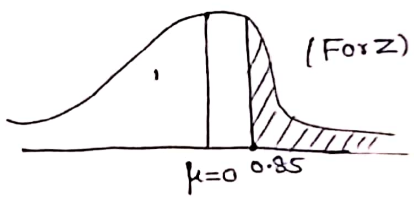
(2) the probability that bulb burns b/w 778 and 834 hrs.

Soln

(1) Given $\mu = 800$, $\sigma = 40$.



Let $z = \dfrac{X-\mu}{\sigma} = \dfrac{834-800}{40} = 0.85$



$P(X > 834) = P(Z > 0.85)$

$= 0.5 - P(Z < 0.85)$

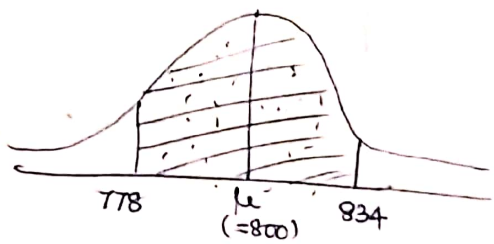
$= 0.5 - 0.0323$ (using Normal table)

$= 0.1977$

(2) Given $X_1 = 778$, $X_2 = 834$

$Z_1 = \dfrac{778-800}{40} = -0.55$

$Z_2 = \dfrac{834-800}{40} = 0.85$



$P(778 < X < 834) = P(-0.55 < Z < 0.85)$

$= P(0 < Z < 0.55) + P(0 < Z < 0.85)$

$= 0.2088 + 0.3023$

$= 0.5111$

(Q) The mean yield for one-acre plot is 662 kilos with standard deviation 32 kilos. Assuming normal distribution, how many one-acre plots in a patch of 1,000 plots would you expect to have yield over 700 kilos and below 650 kilos.

Soln

Given $\mu = 662$, $\sigma = 32$

Standard normal variate $= Z = \frac{X-\mu}{\sigma} = \frac{X-662}{32}$

If $X = 700 \Rightarrow Z = \frac{700-662}{32} = 1.19$

$\therefore P(X>700) = P(Z>1.19)$

$= 0.117$ (using normal table)

$\left(\because P(X>700) = 0.5 - P(Z<1.19)\right.$

$= 0.5 - 0.383$

$\left.= 0.117.\right)$

$\therefore$ The number of plots have yield over 700 kilos $= 117$

If $X = 650 \Rightarrow Z = \frac{650-662}{32} = -0.375$

$\therefore P(X<650) = P(Z<-0.38) = 0.352$

$\therefore$ The number of plots have yield below 650 kilos $= 352$

Problems

① From the following table for bivariate distribution of (X, Y) find

(i) $P(X \le 1)$ (ii) $P(Y \le 3)$, (iii) $P(X \le 1, Y \le 3)$

(iv) $P(X \le 1 / Y \le 3)$ (v) $P(Y \le 3 / X \le 1)$ (vi) $P(X+Y \le 4)$

(vii) the marginal distribution of X (or) marginal PMF of X

(viii) the marginal distribution of Y (or) marginal PMF of Y

(ix) the conditional distribution of X given $Y = 2$

(x) Examine X and Y are independent

(xi) $E(Y - 2X)$

X \ Y	1	2	3	4	5	6
0	0	0	$1/32$	$2/32$	$2/32$	$3/32$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$

Soln

X \ Y	1	2	3	4	5	6	$P_X(x) = f(x)$
0	0 $P(0,1)$	0 $P(0,2)$	$1/32$ $P(0,3)$	$2/32$ $P(0,4)$	$2/32$ $P(0,5)$	$3/32$ $P(0,6)$	$P(X=0) = 8/32$
1	$1/16$ $P(1,1)$	$1/16$ $P(1,2)$	$1/8$ $P(1,3)$	$1/8$ $P(1,4)$	$1/8$ $P(1,5)$	$1/8$ $P(1,6)$	$P(X=1) = 10/16$
2	$1/32$ $P(2,1)$	$1/32$ $P(2,2)$	$1/64$ $P(2,3)$	$1/64$ $P(2,4)$	0 $P(2,5)$	$2/64$ $P(2,6)$	$P(X=2) = 8/64$
$P_Y(y) = f(y)$	$P(Y=1) = 3/32$	$P(Y=2) = 3/32$	$P(Y=3) = 11/64$	$P(Y=4) = 13/64$	$P(Y=5) = 6/32$	$P(Y=6) = 16/64$	Total Propability $=1$.

(i) $P(X \leq 1) = P(X=0) + P(X=1)$

$$= \frac{8}{32} + \frac{10}{16}$$

$$= \frac{8+20}{32}$$

$$= \frac{7}{8}$$

(ii) $P(Y \leq 3) = P(Y=1) + P(Y=2) + P(Y=3)$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64}$$

$$= \frac{6+6+11}{64}$$

$$= \frac{23}{64}$$

(iii) $P(X \le 1, Y \le 3) = P(0,1) + P(0,2) + P(0,3)$
$+ P(1,1) + P(1,2) + P(1,3)$

$= 0 + 0 + \frac{1}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8}$

$= \frac{1+2+2+4}{32}$

$= \frac{9}{32}$

(iv) $P(X \le 1 / Y \le 3) = \frac{P(X \le 1 \;\&\; Y \le 3)}{P(Y \le 3)}$

$= \frac{(9/32)}{(23/64)}$ (using (i) & (ii))

$= \frac{9}{32} \times \frac{64}{23}$

$= \frac{18}{23}$

(v) $P(Y \le 3 / X \le 1) = \frac{P(X \le 1 \;\&\; Y \le 3)}{P(X \le 1)} = \frac{(9/32)}{(7/8)}$

$= \frac{9}{32} \times \frac{8}{7} = \frac{9}{28}$

(vi) $P(X+Y \le 4) = P(0,1) + P(0,2) + P(0,3) + P(0,4) + P(1,1)$
$+ P(1,2) + P(1,3) + P(2,1) + P(2,2)$

$= 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{32} + \frac{1}{32}$

$= \frac{13}{32}$

(viii) The marginal distribution of X is

$P(X=0) = \frac{8}{32}$, $P(X=1) = \frac{20}{32}$, $P(X=2) = \frac{4}{32}$

(viii) The marginal distribution of Y is

$P(Y=1)=\frac{3}{32}$, $P(Y=2)=\frac{3}{32}$, $P(Y=3)=\frac{11}{64}$

$P(Y=4)=\frac{13}{64}$, $P(Y=5)=\frac{6}{32}$, $P(Y=6)=\frac{16}{64}$

(ix) The conditional distribution of X given $Y=2$ is

$P(X=x_i / Y=2)$, $x_i \to 0,1,2$.

$$P(X=0/Y=2)=\frac{P(X=0,Y=2)}{P(Y=2)}=\frac{P(0,2)}{P(Y=2)}=\frac{0}{(3/32)}=0$$

$$P(X=1/Y=2)=\frac{P(X=1,Y=2)}{P(Y=2)}=\frac{P(1,2)}{P(Y=2)}=\frac{(1/16)}{(3/32)}=2/3$$

$$P(X=2/Y=2)=\frac{P(X=2,Y=2)}{P(Y=2)}=\frac{P(2,2)}{P(Y=2)}=\frac{(1/32)}{(3/32)}=1/3$$

(x) Formula for independent

$P(X=i)\times P(Y=j)=P_{ij}$ $\forall i \,\&\, j$.

$\therefore P(X=0)\times P(Y=1)\neq P_{(0,1)}$, X and Y are not independent.

(xi) $E(X)=\sum x_i P(x_i)$ $=0\times P(X=0)+1\times P(X=1)+2\times P(X=2)$

$=(0\times 8/32)+(1\times 20/32)+(2\times 4/32)$

$=0+20/32+8/32$

$=7/8$

$$E(Y) = \sum y_j P(y_j) = [1 \times P(Y=1)] + [2 \times P(Y=2)] + [3 \times P(Y=3)]$$
$$+ [4 \times P(Y=4)] + [5 \times P(Y=5)] + [6 \times P(Y=6)]$$

$$= \left(1 \times 3/32\right) + \left(2 \times 3/32\right) + \left(3 \times 11/64\right) + \left(4 \times 13/64\right) + \left(5 \times 6/32\right)$$
$$+ \left(6 \times 16/64\right)$$

$$= 3/32 + 6/32 + 33/64 + 52/64 + \frac{30}{32} + \frac{96}{64}$$

$$= \frac{6+12+33+52+60+96}{64}$$

$$= \frac{259}{64}.$$

(xii) $$E(Y-2X) = E(Y) - 2E(X)$$

$$= \frac{259}{64} - 2\left(\frac{28}{32}\right)$$

$$= \frac{259-112}{64}$$

$$= \frac{147}{64}.$$

---

② The joint probability mass function of (X, Y) is given by

$P(x, y) = K(2x + 3y)$, $x = 0, 1, 2$
$y = 1, 2, 3$.

Find all the marginal and conditional probability distributions. Also find the probability distn of $(X+Y)$ and $P(X+Y > 3)$

$P(x,y) = K(2x+3y)$

$X \backslash Y$	1	2	3	Total
0	3K	6K	9K	18K
1	5K	8K	11K	24K
2	7K	10K	13K	30K
Total	15K	24K	33K	72K

72K → Total probality

$\Rightarrow 72K = 1$

$\Rightarrow \boxed{K = \frac{1}{72}}$

$X \backslash Y$	1	2	3	
0	$\frac{3}{72}$ P(0,1)	$\frac{6}{72}$ P(0,2)	$\frac{9}{72}$ P(0,3)	$P(X=0) = \frac{18}{72}$
1	$\frac{5}{72}$ P(1,1)	$\frac{8}{72}$ P(1,2)	$\frac{11}{72}$ P(1,3)	$P(X=1) = \frac{24}{72}$
2	$\frac{7}{72}$ P(2,1)	$\frac{10}{72}$ P(2,2)	$\frac{13}{72}$ P(2,3)	$P(X=2) = \frac{30}{72}$
	$P(Y=1) = \frac{15}{72}$	$P(Y=2) = \frac{24}{72}$	$P(Y=3) = \frac{33}{72}$	Total probability $= 1$

The marginal distribution of X :

$P(X=0) = \frac{18}{72}$, $P(X=1) = \frac{24}{72}$, $P(X=2) = \frac{30}{72}$

The marginal destribution of Y :

$P(Y=1) = \frac{15}{72}$, $P(Y=2) = \frac{24}{72}$, $P(Y=3) = \frac{33}{72}$

The conditional distribution of Y, given X is $P(Y=y_j/X=x_i)$

$y_j \to 1,2,3$; $x_i \to 0,1,2$

$$P(Y=1/X=0) = \frac{P(Y=1, X=0)}{P(X=0)} = \frac{(3/72)}{(18/72)} = 1/6$$

$$P(Y=2/X=0) = \frac{P(Y=2, X=0)}{P(X=0)} = \frac{(6/72)}{(18/72)} = 1/3$$

$$P(Y=3/X=0) = \frac{P(Y=3, X=0)}{P(X=0)} = \frac{(9/72)}{(18/72)} = 1/2$$

$$P(Y=1/X=1) = \frac{P(Y=1, X=1)}{P(X=1)} = \frac{(5/72)}{(24/72)} = 5/24$$

$$P(Y=2/X=1) = \frac{P(Y=2, X=1)}{P(X=1)} = \frac{(8/72)}{(24/72)} = 1/3$$

$$P(Y=3/X=1) = \frac{P(X=1, Y=3)}{P(X=1)} = \frac{(11/72)}{(24/72)} = 11/24$$

$$P(Y=1/X=2) = \frac{P(Y=1, X=2)}{P(X=2)} = \frac{(7/72)}{(30/72)} = 7/30$$

$$P(Y=2/X=2) = \frac{P(Y=2, X=2)}{P(X=2)} = \frac{(10/72)}{(30/72)} = 1/3$$

$$P(Y=3/X=2) = \frac{P(X=2, Y=3)}{P(X=2)} = \frac{(13/72)}{(30/72)} = 13/30$$

The conditional distribution of X, given Y is $P\left(X=x_i / Y=y_j\right)$

$x_i \to 0, 1, 2, \quad y_j \to 1, 2, 3.$

$$P\left(X=0/Y=1\right) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{(3/72)}{(15/72)} = 1/5$$

$$P\left(X=1/Y=1\right) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{(5/72)}{(15/72)} = 1/3$$

$$P\left(X=2, Y=1\right) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{(7/72)}{(15/72)} = 7/15$$

$$P\left(X=0/Y=2\right) = \frac{P(X=0, Y=2)}{P(Y=2)} = \frac{(6/72)}{(24/72)} = 1/4$$

$$P\left(X=1/Y=2\right) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{(8/72)}{(24/72)} = 1/3$$

$$P\left(X=2, Y=2\right) = \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{(10/72)}{(24/72)} = 5/12$$

$$P\left(X=0, Y=3\right) = \frac{P(X=0, Y=3)}{P(Y=3)} = \frac{(9/72)}{(33/72)} = 9/33 = 3/11.$$

$$P\left(X=1, Y=3\right) = \frac{P(X=1, Y=3)}{P(Y=3)} = \frac{(11/72)}{(33/72)} = 1/3.$$

$$P\left(X=2, Y=3\right) = \frac{P(X=2, Y=3)}{P(Y=3)} = \frac{(13/72)}{(33/72)} = 13/33$$

Now $P\left(X+Y > 3\right) = P(X+Y=4) + P(X+Y=5)$

$$= P(1,3) + P(2,2) + P(2,3)$$

$$= \frac{11}{72} + \frac{10}{72} + \frac{13}{72}$$

$$= \frac{34}{72}$$

③ Suppose the joint pdf is given by (A)

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & ; 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & ; \text{otherwise} \end{cases}$$

Obtain the marginal pdf of X and that of Y. Hence find $P\left[\frac{1}{4} \le y \le \frac{3}{4}\right]$.

Soln

Given $f(x,y) = \begin{cases} 6/5\,(x+y^2) & ; 0 \le x \le 1;\ 0 \le y \le 1 \\ 0 & ; \text{otherwise} \end{cases}$

The marginal p.d.f of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{6}{5}(x+y^2)\,dy$$

$$= \frac{6}{5}\left\{xy + y^3/3\right\}_0^1$$

$$= 6/5\left\{x + 1/3\right\} \quad ; 0 \le x \le 1.$$

The marginal p.d.f of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{6}{5}(x+y^2)\,dx$$

$$= \frac{6}{5}\left(x^2/2 + y^2x\right)_0^1$$

$$= 6/5\left(x^2/2 + y^2x\right)_0^1$$

$$= 6/5\left(1/2 + y^2\right) \quad ; \quad 0 \le y \le 1.$$

$$P\left[1/4 \le y \le 3/4\right] = \int_{1/4}^{3/4} f(y)\,dy = 6/5 \int_{1/4}^{3/4} (1/2 + y^2)\,dy$$

$$= 6/5\left[y/2 + y^3/3\right]_{1/4}^{3/4} = \frac{6}{5}\left[3/8 + \frac{27}{192} - 1/8 - 1/192\right]$$

$$= 6/5\left[2/8 - 26/192\right] = 6/5\left[\frac{48-26}{192}\right] = 6/5 \times \frac{\overset{11}{\cancel{22}}}{\underset{\underset{16}{\cancel{96}}}{\cancel{192}}} = 11/80.$$

4. Let X and Y have joint pdf $f(x,y) = 2,\ 0 < x < y < 1$. Find the marginal density function and find the conditional density function of Y given $X = x$.

Soln

The marginal density function of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

Here y varies from x to 1.

$$\therefore f(x) = \int_{x}^{1} 2\, dy = 2(1-x). \quad ; \ 0 < x < 1.$$

The marginal density function of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_{0}^{y} 2\, dx = 2y.$$

The conditional density function of Y given $X = x$ is

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{2}{2(1-x)} = (1-x)^{-1} \ ; \ 0 < x < 1.$$

5. The joint p.d.f of the random variable (X, Y) is given by

$$f(x,y) = K xy e^{-(x^2+y^2)}, \quad x > 0,\ y > 0.$$

Find the value of K and also prove that X and Y are independent.

Soln

Given $f(x,y) = K xy e^{-(x^2+y^2)}, \quad x > 0, y > 0$

$\therefore$ x varies from 0 to ∞

y varies from 0 to ∞.

$$\therefore \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_{0}^{\infty}\int_{0}^{\infty} Kxy\,e^{-(x^2+y^2)}\,dx\,dy = 1.$$

$$\Rightarrow K \times \int_{0}^{\infty} x e^{-x^2} dx \times \int_{0}^{\infty} y e^{-y^2} dy = 1 \longrightarrow (1)$$

Take $\int_{0}^{\infty} x e^{-x^2} dx$

Put $x^2 = t$

$\Rightarrow 2x\,dx = dt$

$\Rightarrow x\,dx = dt/2$

When $x = 0$, $t = 0$

When $x = \infty$, $t = \infty$.

$$\therefore \int_{0}^{\infty} x e^{-x^2} dx = \int_{0}^{\infty} e^{-t} dt/2 = \frac{1}{2}\left(\frac{e^{-t}}{-1}\right)_{0}^{\infty} = \frac{1}{2}\left(0 - \frac{1}{-1}\right)$$

$$= 1/2.$$

$$\therefore \int_{0}^{\infty} x e^{-x^2} dx = \int_{0}^{\infty} y e^{-y^2} dy = 1/2 \longrightarrow (2)$$

$\therefore$ From (1) $\quad K \times 1/2 \times 1/2 = 1$

$$\Rightarrow \boxed{K = 4}$$

To prove: X and Y are independent.

It is enough to prove : $f(x,y) = f(x) \times f(y)$

$f(x)$ = marginal density of X

$$= \int_{0}^{\infty} Kxy\,e^{-(x^2+y^2)} dy = 4\int_{0}^{\infty} xy \cdot e^{-x^2} e^{-y^2} dy$$

$$= 4xe^{-x^2}\int_{0}^{\infty} ye^{-y^2} dy$$

$$= 4x\,e^{-x^2} \times 1/2$$

$$= 2xe^{-x^2} \; ; \; 0 < x < \infty.$$

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$||^{ly}$ the marginal ~~pdf~~ density of Y is given by

$$f(y) = \int_0^\infty f(x,y)\, dx$$

$$= \int_0^\infty 4xy\, e^{-x^2} \cdot e^{-y^2} dx$$

$$= 4y e^{-y^2} \int_0^\infty x \cdot e^{-x^2} dx$$

$$= 4y e^{-y^2} \times 1/2$$

$$= 2y e^{-y^2} \quad ; 0 < y < \infty.$$

Now consider $f(x) \times f(y) = 2x e^{-x^2} \times 2y e^{-y^2}$

$$= 4xy\, e^{-(x^2+y^2)}$$

$$= f(x,y)$$

$\therefore$ The random Variables X and Y are independent.

(6) Given $f_{xy}(x,y) = \begin{cases} Cx(x-y) & ; \quad 0 < x < 2 \\ & \quad -x < y < x \\ 0 & ; \quad \text{otherwise.} \end{cases}$

(a) Evaluate C (b) Find $f_X(x)$ (c) $f_{Y/X}(y/x)$

and (d) $f_Y(y)$.

Soln

W.K.T $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1.$

$$\Rightarrow \int_0^2 \int_{-x}^{x} Cx(x-y)\, dy\, dx = 1.$$

$$\Rightarrow C\int_0^2 \left(x^2 y - \frac{y^2}{2} x\right)_{y=-x}^{y=x} dx = 1.$$

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$$\Rightarrow C \int_0^2 \left(x^3 - x^3/2 + x^3 + x^3/2 \right) dx = 1.$$

$$\Rightarrow C \int_0^2 2x^3 \, dx = 1$$

$$\Rightarrow C \left[2x^4/4 \right]_0^2 = 1$$

$$\Rightarrow C \left[x^4/2 \right]_0^2 = 1$$

$$\Rightarrow C \left[8 \right] = 1$$

$$\Rightarrow \boxed{C = 1/8}$$

(b) The marginal density of X is given by

$$f(x) = \int_{-x}^{x} Cx(x-y)\,dy = \int_{-x}^{x} \frac{1}{8}(x^2 - xy)\,dy$$

$$= \cancel{\frac{1}{8}\left[\frac{x^3}{3}\right]} = \frac{1}{8}\left[x^2 y - xy^2/2\right]_{y=-x}^{y=x}$$

$$= 1/8 \left[x^3 - x^3/2 + x^3 + x^3/2 \right]$$

$$= 1/8 \left[2x^3 \right]$$

$$= x^3/4 \quad ; \quad 0 < x < 2$$

(c) $f\left(\frac{y}{x}\right) = \frac{f(x,y)}{f(x)} = \frac{\frac{1}{8}x(x-y)}{(x^3/4)}$

$= \frac{1}{2x^2}(x-y) \quad ; \quad -x < y < x.$

(d) $f(y) = \begin{cases} \int_{-y}^{2} \frac{1}{8} x(x-y)\,dx & \text{if } -2 \le y \le 0 \\ \int_{y}^{2} \frac{1}{8} x(x-y)\,dx & \text{if } 0 \le y \le 2 \end{cases}$

$= \begin{cases} \frac{1}{8}\int_{-y}^{2} (x^2 - xy)\,dx \\ \frac{1}{8}\int_{y}^{2} (x^2 - xy)\,dx \end{cases} = \begin{cases} \frac{1}{8}\left[x^3/3 - x^2 y/2\right]_{x=-y}^{x=2} \\ 1/8\left[x^3/3 - x^2 y/2\right]_{x=y}^{x=2} \end{cases}$

$= \begin{cases} \frac{1}{8}\left[(8/3 - 2y) - \left(-y^3/3 - y^3/2\right)\right] \\ 1/8\left[(8/3 - 2y) - \left(y^3/3 - y^3/2\right)\right] \end{cases}$

$= \begin{cases} 1/3 - y/4 + 5/48\, y^3 & \text{if } -2 \le y \le 0 \\ 1/3 - y/4 + 1/48\, y^3 & \text{if } 0 \le y \le 0. \end{cases}$

Convariance, Correlation and Regression

Covariance: If X and Y are random variables, then co-variance b/w them is defined as

$$\boxed{cov(X,Y) = E(XY) - E(X).E(Y)}$$

* If X and Y are independent, then $cov(X,Y) = 0$

  ⇒ If X and Y are independent then $E(XY) = E(X).E(Y)$

* $cov(aX, bY) = ab\ cov(X,Y)$

* $cov(X+a, Y+b) = cov(X,Y)$

* $cov(aX+b, cY+d) = ac\ cov(X,Y)$

* $V(X_1+X_2) = V(X_1) + V(X_2) + 2\,cov(X_1, X_2)$

* $V(X_1-X_2) = V(X_1) + V(X_2) - 2\,cov(X_1, X_2)$

Correlation:

Let X and Y be given random variables.

Then Karl pearson's coefficient of correlation is denoted by

$\gamma(X,Y)$ or γ_{XY} and defined as

$$\boxed{\gamma(X,Y) = \gamma_{XY} = \frac{cov(X,Y)}{\sigma_X . \sigma_Y}}$$

where $cov(X,Y) = E(XY) - E(X).E(Y)$

$$\sigma_x^2 = \frac{\sum x^2}{n} - (\bar{x})^2$$

$$\sigma_y^2 = \sum y^2/n - (\bar{y})^2$$

* Coefficient of correlation always lies b/w -1 to $+1$

* Two random variables with non-zero correlation are said to be correlated.

Rank correlation:

Suppose a group of 'n' individuals is arranged in order of merit or proficiency of possession of two characteristics A and B.

If (x_i, y_i) $i = 1, 2, \ldots, n$ are the ranks of the individuals in characteristics A and B respectively, then the rank correlation coefficient is given by

$$\boxed{r = 1 - \frac{6}{n(n^2-1)} \sum_{i=1}^{n} (x_i - y_i)^2}$$

* If any two or more individuals are equal in any classification w.r.t characteristic A or B then we add the correction factor $\frac{m(m^2-1)}{12}$ to $\sum (x_i - y_i)^2$

where $m \rightarrow$ number of times an item is repeated.

This correction factor is to be added for each repeated value.

Problems

① Calculate the correlation - coefficient for the following heights (in inches) of fathers X and their sons Y.

X:	65	66	67	67	68	69	70	72
Y:	67	68	65	68	72	72	69	71.

Soln

$n = 8$

	X	Y	XY	X^2	Y^2
	65	67	4355	4225	4489
	66	68	4488	4356	4624
	67	65	4355	4489	4225
	67	68	4556	4489	4624
	68	72	4896	4624	5184
	69	72	4968	4761	5184
	70	69	4830	4900	4761
	72	71	5112	5184	5041
Total	544	552	37560	37028	38132

$\bar{x} = \Sigma x/n = 544/8 = 68$

$\bar{y} = \Sigma y/n = 552/8 = 69$

$\sigma_x^2 = \Sigma x^2/n - (\bar{x})^2 = \frac{37028}{8} - 68^2 \Rightarrow \sigma_x = 2.121.$

$\sigma_y^2 = \Sigma y^2/n - (\bar{y})^2 = 38132/8 - 69^2 \Rightarrow \sigma_y = 2.345$

$\mathrm{cov}(x,y) = \frac{\Sigma xy}{n} - \bar{x}\cdot\bar{y} = \frac{37560}{8} - (68 \times 69) = 3$

The correlation coefficient $= r_{xy} = \frac{\mathrm{cov}(x,y)}{\sigma_x \cdot \sigma_y} = \frac{3}{(2.121 \times 2.345)}$

$= 0.6032$

② Find the rank correlation coefficient from the following data

Rank in X	1	2	3	4	5	6	7
Rank in Y	4	3	1	2	6	5	7.

Soln

X	Y	$d_i=(x_i-y_i)$	d_i^2
1	4	−3	9
2	3	−1	1
3	1	2	4
4	2	2	4
5	6	−1	1
6	5	1	1
7	7	0	0

Here $n=7$.

$$\Sigma d_i^2 = 20.$$

$\therefore$ Rank correlation coefficient $= r = 1-\frac{6 \Sigma d_i^2}{n(n^2-1)}$

where $d_i = (x_i-y_i)$

$$\Rightarrow r = 1-\frac{6(20)}{7(7^2-1)}$$

$$= 1-\frac{6(20)}{7\times 48}$$

$$= 1-20/56$$

$$= 0.6429.$$

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(3) Obtain the rank correlation coefficient for the following data.

X	68	64	75	50	64	80	75	40	55	64
Y	62	58	68	45	81	60	68	48	50	70

Soln

X	Y	Rank (X) (x_i)	Rank(Y) (y_i)	$d_i = (x_i - y_i)$	d_i^2
68	62	4	5	-1	1
64	58	6	7	-1	1
75	68	2.5	3.5	-1	1
50	45	9	10	-1	1
64	81	6	1	5	25
80	60	1	6	-5	25
75	68	2.5	3.5	-1	1
40	48	10	9	1	1
55	50	8	8	0	0
64	70	6	2	4	16

$\Sigma d_i^2 = 72$.

Correction factors

In X column, 75 is repeated twice ($m=2$)

$\therefore \boxed{C.F = \frac{m(m^2-1)}{12}} \Rightarrow \frac{2(2^2-1)}{12} = \boxed{1/2 = C.F_1}$

Also in X column, 64 is repeated thrice ($m=3$)

$\therefore C.F_2 = \frac{3(8)}{12} = 2$

In Y Series, 68 repeated twice ($m=2$)

$\Rightarrow C.F_3 = 1/2$

$$\therefore r = 1 - \frac{6}{n(n^2-1)}\left[\Sigma d_i^2 + CF_1 + CF_2 + CF_3\right]$$

$$= 1 - \frac{6}{10(10^2-1)}\left[72 + 1/2 + 2 + 1/2\right]$$

$$= 0.5454$$

(4) Suppose that the 2 dimensional random variables (X, Y) has the joint p.d.f

$$f(x,y) = \begin{cases} x+y, & 0 < x < 1,\ 0 < y < 1 \\ 0; & \text{Otherwise} \end{cases}$$

(i) Obtain the correlation co-efficient b/w X and Y

(ii) Check whether X and Y are independent.

Soln

The marginal density function of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_{y=0}^{y=1} = x + \frac{1}{2}$$

The marginal density function of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (x+y)\,dx = \left(\frac{x^2}{2} + yx\right)_{x=0}^{x=1} = \frac{1}{2} + y.$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^1 x\left(x + \frac{1}{2}\right) dx = \left(\frac{x^3}{3} + \frac{x^2}{4}\right)_0^1$$
$$= \frac{7}{12}.$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f(y)\,dy = \int_0^1 y\left(y + \frac{1}{2}\right) dy = \left(\frac{y^3}{3} + \frac{y^2}{4}\right)_0^1 = \frac{7}{12}.$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^1 x^2\left(x + \frac{1}{2}\right) dx = \left(\frac{x^4}{4} + \frac{x^3}{6}\right)_0^1 = \frac{5}{12}.$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f(y)\,dy = \frac{5}{12}.$$

Now $\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$

$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2 = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}.$

$\Rightarrow \sigma_x = \frac{\sqrt{11}}{12}, \quad \sigma_y = \frac{\sqrt{11}}{12}$

$\therefore$ Correlation coefficient $= \gamma_{xy} = \frac{\text{Cov}(x,y)}{\sigma_x \cdot \sigma_y}$

where $\text{Cov}(x,y) = E(xy) - E(x).E(y)$.

Now $E(xy) = \int_0^1 \int_0^1 xy(x+y)\,dx\,dy$

$= \int_0^1 \int_0^1 (x^2y + xy^2)\,dx\,dy$

$= \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_0^1 dy$

$= \int_0^1 \left(\frac{y}{3} + \frac{y^2}{2}\right) dy$

$= \left(\frac{y^2}{6} + \frac{y^3}{6}\right)_0^1$

$= \frac{2}{6}.$ $\quad \therefore \text{Cov}(x,y) = E(xy) - E(x).E(y)$

$= \frac{2}{6} - \frac{49}{144} = \frac{-1}{144}$

$\gamma_{xy} = \frac{\text{Cov}(x,y)}{\sigma_x . \sigma_y} = \frac{(-1/144)}{(11/144)} = -\frac{1}{11} = -0.0909$

$\therefore$ correlation coefficient $= \gamma_{xy} = -\frac{1}{11}$.

Since $\text{Cov}(x,y) \neq 0$, X & Y are not independent.

(6) Two independent random variables X and Y are defined by

$f(x) = 4ax \ , 0 \le x \le 1$
$\quad = 0 \ ; \text{o.w}$

$f(y) = 4by \ ; 0 \le y \le 1$
$\quad = 0 \ ; \text{o.w}$

Show that $U = X+Y$ and $V = X-Y$ are uncorrelated.

Soln

W.K.T $\int_{-\infty}^{\infty} f(x).dx = 1$ ($\because f(x) \to$ probability density function of X)

$\int_0^1 4ax.dx = 1 \Rightarrow 4a\left(\frac{x^2}{2}\right)_0^1 = 1 \Rightarrow 4a \times \frac{1}{2} = 1$

$\Rightarrow \boxed{a = \frac{1}{2}}$

Similarly $\int_0^1 4by.dy = 1 \Rightarrow 4b\left(\frac{y^2}{2}\right)_0^1 = 1 \Rightarrow \boxed{b = \frac{1}{2}}$

To prove U and V are uncorrelated.

It is enough to prove $\text{Cov}(U,V) = 0$.

$E(X) = \int_0^1 x f(x).dx = \int_0^1 x\,4ax.dx = \int_0^1 2x^2 dx = \frac{2}{3}$.

$E(Y) = \int_0^1 x f(y)\,dy = \int_0^1 y.4by\,dy = \int_0^1 2y^2 dy = \frac{2}{3}$.

Given that X and Y are independent

$\Rightarrow E(XY) = E(X).E(Y)$

$\Rightarrow E(XY) = \frac{4}{9}$.

$E(U) = E(X+Y) = E(X)+E(Y) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$.

$E(V) = E(X-Y) = E(X)-E(Y) = \frac{2}{3} - \frac{2}{3} = 0$.

$E(UV) = E\left((X+Y).(X-Y)\right) = E\left(X^2-Y^2\right) = E(X^2)-E(Y^2) \longrightarrow (1)$.

Now $E(X^2) = \int_0^1 x^2\,4ax.dx = 2\int_0^1 x^3 dx = 2\left(\frac{x^4}{4}\right)_0^1 = 2\left(\frac{1}{4}\right) = \frac{1}{2}$

Also $E(Y^2) = \frac{1}{2}$. $\therefore$ From (1), $E(UV) = 0$.

$\text{Cov}(U,V) = E(UV) - E(U).E(V) = 0 - \frac{4}{3}(0) = 0$.

$\therefore$ U & V are uncorrelated

(6) Two random variables X and Y have the joint p.d.f

$$f(x,y) = \begin{cases} 2-x-y \; ; & 0<x<1,\ 0<y<1 \\ 0 \; ; & \text{otherwise} \end{cases}$$

Then show that $\text{cov}(X,Y) = -1/144$

Soln

Given $f(x,y) = \begin{cases} 2-x-y \; ; & 0<x<1,\ 0<y<1 \\ 0 \; ; & \text{otherwise} \end{cases}$

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (2-x-y)$$

$$= \left[(2-x)y - \frac{y^2}{2}\right]_{y=0}^{y=1}$$

$$= (2-x) - 1/2$$

$$\boxed{f(x) = 3/2 - x}, \quad 0<x<1$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (2-x-y)\,dx$$

$$= \left[(2-y)x - \frac{x^2}{2}\right]_{x=0}^{x=1}$$

$$= (2-y) - 1/2$$

$$\boxed{f(y) = 3/2 - y}; \quad 0<y<1$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x(3/2 - x)\,dx$$

$$= \int_0^1 (3/2\,x - x^2)\,dx$$

$$= \left(\frac{3}{4}\cdot x^2 - \frac{x^3}{3}\right)_0^1$$

$$= 3/4 - 1/3$$

$$\boxed{E(X) = \frac{5}{12}}$$

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$$E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy$$

$$= \int_0^1 y\left(\tfrac{3}{2} - y\right) dy$$

$$= \int_0^1 \left(\tfrac{3}{2}y - y^2\right) dy$$

$$= \left(\frac{3}{4}y^2 - \frac{y^3}{3}\right)_{y=0}^{y=1}$$

$$= \frac{3}{4} - \frac{1}{3}$$

$$\boxed{E(Y) = \frac{5}{12}}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 xy(2-x-y)\, dx\, dy$$

$$= \int_0^1\int_0^1 (2xy - x^2y - xy^2)\, dx\, dy$$

$$= \int_0^1 \left(\frac{2x^2y}{2} - \frac{x^3y}{3} - \frac{x^2y^2}{2}\right)_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{2y}{2} - \frac{y}{3} - \frac{y^2}{2}\right) dy$$

$$= \left(\frac{2y^2}{4} - \frac{y^2}{6} - \frac{y^3}{6}\right)_{y=0}^{y=1}$$

$$= \frac{2}{4} - \frac{1}{6} - \frac{1}{6}$$

$$= \frac{6-2-2}{12}$$

$$\boxed{E(XY) = \frac{1}{6}}$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)\,E(Y)$$

$$= \frac{1}{6} - \left(\frac{5}{12} \times \frac{5}{12}\right)$$

$$= \frac{1}{6} - \frac{25}{144}$$

$$= \frac{24-25}{144}$$

$$\mathrm{Cov}(X,Y) = -\frac{1}{144}$$

Regression: Regression is a mathematical measure of the average relationship b/w two or more variables in terms of the original limits of the data.

Lines of regression:

(1) $y - \bar{y} = r \dfrac{\sigma_y}{\sigma_x}(x - \bar{x})$ is regression line of y on x.

(2) $x - \bar{x} = r \dfrac{\sigma_x}{\sigma_y}(y - \bar{y})$ is regression line of x on y.

Notes

* Both the lines of regression passes through $(\bar{x}, \bar{y})$

* Regression coefficient of y on x $= r\dfrac{\sigma_y}{\sigma_x} = b_{yx}$

  Regression coefficient of x on y $= r\dfrac{\sigma_x}{\sigma_y} = b_{xy}$.

* Correlation coefficient $= r = \pm\sqrt{b_{yx} \cdot b_{xy}}$

  Where $b_{yx} = \dfrac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2}$

  $b_{xy} = \dfrac{\sum (x-\bar{x})(y-\bar{y})}{\sum (y-\bar{y})^2}$

Properties of Regression lines:

(1) The regression lines pass through $(\bar{x}, \bar{y})$

∴ $(\bar{x}, \bar{y})$ is the point of intersection of the regression lines.

(2) When $r = 1$, perfect +ve correlation.

$r = -1$, perfect −ve correlation.

(3) When $r = 0$, the equation of the regression lines are $y = \bar{y}$, and $x = \bar{x}$ which represent $\perp^r$ lines which are parallel to the axes.

(4) The slopes of the lines are $r\frac{\sigma_y}{\sigma_x}$, $\frac{1}{r}\frac{\sigma_y}{\sigma_x}$.

(5) Angle b/w regression lines

The slopes of the regression lines are

$$m_1 = r\frac{\sigma_y}{\sigma_x}, \quad m_2 = \frac{1}{r}\frac{\sigma_y}{\sigma_x}.$$

$\theta \to$ angle b/w reg. line

$$\Rightarrow \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\left(\frac{1-r^2}{r}\right).$$

(i) When $r = 0$, no correlation b/w x & y.

(ii) When $\theta = \pi/2$, reg. lines are $\perp^r$

(iii) When $r = 1$ (or) $r = -1$, there is a perfect correlation, +ve or −ve, $\theta = 0$ and so the lines coincide.

(6) Correlation coefficient is geometric mean b/w the two regression coefficients.

$$r = \pm\sqrt{b_{yx} \cdot b_{xy}}.$$

(7) If one of the regression coefficient is greater than unity then other is less than unity.

Problems

① From the following data, find

(i) the two regression equations

(ii) the coefficient of correlation b/w marks in economics & Statistics

(iii) the most likely marks in Statistics when marks in economics are 30.

Marks in Economics (x) : 25 28 35 32 31 36 29 38 34 32

Marks in Statistics (y) : 43 46 49 41 36 32 31 30 33 39.

Soln

x	y	$(x-\bar{x})$	$(y-\bar{y})$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
25	43	-7	5	49	25	-35
28	46	-4	8	16	64	-32
35	49	3	11	9	121	33
32	41	0	3	0	9	0
31	36	-1	-2	1	4	2
36	32	4	-6	16	36	-24
29	31	-3	-7	9	49	21
38	30	6	-8	36	64	-48
34	33	2	-5	4	25	-10
32	39	0	1	0	1	0

$\bar{x} = \frac{\Sigma x}{n} = \frac{320}{10} = 32$; $\bar{y} = \frac{\Sigma y}{n} = \frac{380}{10} = 38$

$b_{yx} = \frac{\Sigma (x-\bar{x})(y-\bar{y})}{\Sigma (x-\bar{x})^2} = -93/140 = -0.6643$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2} = \frac{-93}{398} = -0.2337.$$

(i) Eqn of regression line of x on y

$$(x-\bar{x}) = b_{xy}(y-\bar{y})$$

$$(x-32) = -0.2337(y-38)$$

$$x = -0.2337y + 40.8806.$$

Eqn of the line of regression of y on x.

$$y-\bar{y} = b_{yx}(x-\bar{x})$$

$$y-38 = -0.6643(x-32)$$

$$y = -0.6643x + 59.2576$$

(ii) Coefficient of correlation

$$r^2 = b_{yx} \cdot b_{xy} = -0.6643 \times -0.2337$$

$$= 0.1552$$

$$\therefore r = \sqrt{0.1552} = \pm 0.394.$$

(iii) The most likely marks in statistics (y) when marks in ~~statistics~~ Economics (x) are 30.

Put $x=30$ in regression line of y on x.

$$\therefore y = (-0.6643(x) + 59.2576)$$

Put $x=30$

$$y = 39.$$

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② The two lines of regression

$$8x - 10y + 66 = 0 \longrightarrow (A)$$

$$40x - 18y - 214 = 0 \longrightarrow (B)$$

The variance of x is 9 (i.e., $V(x) = 9$)

Find (i) the mean values of x and y

(ii) Correlation coefficient b/w x and y.

Soln

(i) W.K.T both the lines of regression pass through $(\bar{x}, \bar{y})$ where $\bar{x} \rightarrow$ mean value of x

$\bar{y} \rightarrow$ mean value of y.

$$\therefore \quad 8\bar{x} - 10\bar{y} = -66 \longrightarrow ①$$

$$40x - 18y = 214 \longrightarrow ②$$

$$① \times 5 \Rightarrow \quad 40\bar{x} - 50\bar{y} = -330$$

$$② \Rightarrow \quad \overset{(-)}{40\bar{x}} \overset{(+)}{-} 18\bar{y} \overset{(-)}{=} 214$$

$$(-) \qquad -32\bar{y} = -544$$

$$\bar{y} = \frac{-544}{-32} = 17.$$

Sub $\bar{y} = 17$ in ①, $\quad 8\bar{x} = -66 + 10(17) = 104$

$$\bar{x} = 13$$

$\therefore$ Mean value of $x = \bar{x} = 13$

Mean value of $y = \bar{y} = 17$.

(ii) To find the correlation coefficient b/w x and y.

(A) may represent either reg. line of y on x

or reg. line of x on y.

From (A) $\Rightarrow$ $8x = 10y - 66$ (or) $10y = 8x + 66$.

$x = \frac{10}{8}y - \frac{66}{8}$ $\qquad y = \frac{8}{10}x + \frac{66}{10}$.

(i) $b_{xy} = \frac{10}{8}$. $\qquad b_{yx} = 8/10$.

11ly From (B) $\Rightarrow$

$18y = 40x - 214$ (or) $40x = 18y + 214$

$y = \frac{40}{18}x - \frac{214}{18}$ $\qquad x = \frac{18}{40}y + \frac{214}{40}$

$b_{yx} = \frac{40}{18}$ $\qquad b_{xy} = 18/40$.

$\Rightarrow r = \sqrt{\frac{10}{8} \times \frac{40}{18}}$ $\qquad \Rightarrow r = \sqrt{8/10 \times 18/40}$

$r = \sqrt{2.77}$ $\qquad = \pm 0.6$.

$r = 1.66$ $\qquad$ (possible)

(is not possible)

$\therefore$ $r = \pm(0.6)$ ~~(may be)~~ (only possible case)

Since both the reg. coefficients are +ve, r also +ve.

$\therefore \boxed{r = 0.6}$

Regression coefficient $= r = 0.6$.

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(3) For the following data find the most likely price at Madras corresponding to the price 70 at Bombay and that at Bombay corresponding to the price 68 at Madras.

	Madras	Bombay
Average price	65	67
S.D of price	0.5	3.5

(S.D of the difference b/w the price at Madras & Bombay is 3.1)

Soln

Let x denote the price at Madras

y denote the price at Bombay.

Given $\bar{x} = 65$, $\bar{y} = 67$.

$\sigma_x = 0.5$, $\sigma_y = 3.5$, $\sigma_{x-y} = 3.1$

Correlation coefficient $= r = \dfrac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x \sigma_y}$

$$= \frac{0.25 + 12.25 - 9.61}{2 \times 0.5 \times 3.5} = 0.83$$

The reg. line of y on x is

$$y - \bar{y} = r \cdot \frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

$$y - 67 = \frac{0.83 \times 3.5}{0.5}(x - 65).$$

Put $x = 68$,

$$y - 67 = \frac{0.83 \times 3.5}{0.5} \times 3 \Rightarrow y = 84.43$$

$\therefore$ corresponding to the price 68 at Madras, the most likely price at Bombay is 84.43

Similarly, the line of regression of x on y is

$$(x-\bar{x}) = r \cdot \frac{\sigma_x}{\sigma_y} (y-\bar{y})$$

$$x-65 = 0.83 \times \frac{0.5}{3.5} (y-67).$$

Put $y=70$.

$$x-65 = \frac{0.83 \times 0.5}{3.5} (3)$$

$$\Rightarrow x = 65.36.$$

$\therefore$ corresponding to the price 70 at Bombay, the most likely price at Madras is 65.36.

④ The regression eqn of X on Y is $3Y-5X+108=0$. If the mean value of Y is 44 and the variance of X is $\left(\frac{9}{16}\right)^{th}$ of the variance of Y. Find the mean value of X and the correlation co-efficient.

Soln

Given $\bar{y} = 44$.

$$3\bar{y} - 5\bar{x} = -108. \longrightarrow ①$$

$$\Rightarrow 3(44) - 5\bar{x} = -108$$

$$\Rightarrow 5\bar{x} = 240 \quad \Rightarrow \boxed{\bar{x} = 48}$$

① $\Rightarrow$

$$5\bar{x} = 3\bar{y} + 108 \quad \Rightarrow \bar{x} = \frac{3}{5}\bar{y} + \frac{108}{5}$$

$$\Rightarrow b_{xy} = r\frac{\sigma_x}{\sigma_y} = \frac{3}{5}. \longrightarrow ②$$

Given $\sigma_x^2 = \frac{9}{16}\sigma_y^2 \Rightarrow \frac{\sigma_x}{\sigma_y} = \frac{3}{4}. \rightarrow ③$

Sub ③ in ②, $b_{xy} = r\left(\frac{3}{4}\right) = \frac{3}{5}$

$$\Rightarrow r = \frac{4}{5} = 0.8$$

Transformation of Random Variables

If (X,Y) is two dimensional random variable with joint p.d.f $f_{XY}(x,y)$ and if $Z = g(X,Y)$ and $W = h(X,Y)$ are two other random variables then the joint p.d.f of (Z,W) is given by

$$f_{ZW}(z,w) = \frac{f_{XY}(x,y)}{|J|} \quad \text{where } J = \frac{\partial(z,w)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix}$$

(OR) $\boxed{f(x,y) = |J| \cdot f(z,w)}$

Note

* This result holds good, only if the eqn $Z = g(X,Y)$ and $W = h(X,Y)$ when solved, give unique values of x and y in terms of z and w.

One Function of two random Variables

If a random variable Z is defined as $Z = g(X,Y)$ where X and Y are given random variables with joint pdf $f(x,y)$.

To find the p.d.f of Z, we introduce a second random variable $W = h(X,Y)$ and obtain the joint pdf of (Z,W) by using the previous result. Let it be $f_{ZW}(z,w)$.

The required p.d.f of Z is then obtained as the marginal p.d.f of $f_Z(z)$ is obtained by simply integrating $f_{ZW}(z,w)$ w.r.t 'w'.

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)\, dw$$

Problems

① If X and Y are independent random variables with p.d.f $e^{-x}, x \geq 0$ and $e^{-y}; y \geq 0$ respectively. Find the density function of $U = \frac{X}{X+Y}$, and $V = X+Y$.

Are U and V are independent ?

Soln

Since X and Y are independent

$$f(x,y) = f(x) \cdot f(y) = e^{-(x+y)} \; ; \; x,y \geq 0.$$

Given $u = \frac{x}{x+y} \rightarrow ①$ | $v = x+y. \rightarrow ②.$

Sub ② in ①

$$u = \frac{x}{v}$$

$$x = uv \rightarrow ③$$

$$y = v - x$$

$$y = v - uv \quad (\because \text{using ③}) \rightarrow ④.$$

$\therefore \quad x = uv \quad \& \quad y = v - uv.$

$$\frac{\partial x}{\partial u} = v \qquad \frac{\partial y}{\partial u} = -v$$

$$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = 1-u.$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v - uv + uv = v.$$

$\therefore$ $f(x,y) \times |J| = f(u,v).$

$$\Rightarrow f(u,v) = f(x,y) \times |J| = v\, e^{-v}.$$

The density function of U is

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\, dv$$

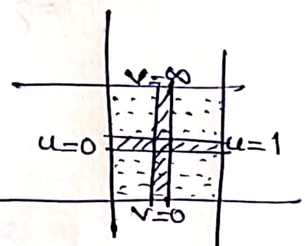
$$= \int_{0}^{\infty} v e^{-v}\, dv$$

$$= \left[v \cdot \frac{e^{-v}}{(-1)} - (1) \cdot \frac{e^{-v}}{1^2} \right]_{v=0}^{v=\infty}$$

$$= \left[(0-0) - (0-1) \right]$$

$$f(u) = 1.$$

Limit Diagram



$0 \le u \le 1.$

$0 \le v \le \infty.$

The density function of V is

$$f(v) = \int_{0}^{1} v e^{-v}\, du = v e^{-v}.$$

To check U and V are independent.

If $f(u,v) = f(u) \times f(v)$ then we know that U and V are independent.

$$v e^{-v} = 1 \times v e^{-v}$$

$\therefore$ U and V are independent.

(2) If X and Y are independent exponential distn with parameter 1 then find the p.d.f of $U = X - Y$.

Soln

p.d.f of $X = e^{-x}$; $x > 0$

p.d.f of $Y = e^{-y}$; $y > 0$.

Since X and Y are independent, joint pdf $= e^{-(x+y)}, \; x, y > 0$.

Given $u = x - y$ | $v = y$.

$u = x - v$.

$x = u + v$

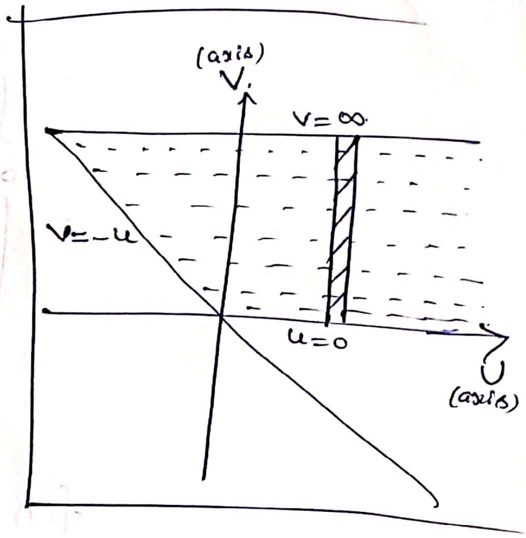
$\cancel{\Rightarrow y = x - u}$

$\cancel{y}$

$$\Rightarrow \quad \frac{\partial x}{\partial u} = 1, \quad \frac{\partial x}{\partial v} = 1$$

$$\frac{\partial y}{\partial u} = 0, \quad \frac{\partial y}{\partial v} = 1.$$

Lemit Diagram



$x > 0$ | $y > 0$.

$u + v > 0$. | $v > 0$.

$u > -v$

$$\therefore \; J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1.$$

$$f(u,v) = |J| \cdot f(x,y) \Rightarrow f(u,v) = f(x,y) = e^{-(u+2v)}$$

Case (i) If $u < 0$ then

$$f(u) = \int_{-u}^{\infty} e^{-(u+2v)} \, dv$$

$$\Rightarrow f(u) = e^{-u} \cdot \left(\frac{e^{-2v}}{-2} \right)_{v=-u}^{v=\infty} = e^{-u} \left(0 - \frac{e^{+2u}}{-2} \right) = e^{u}/2$$

Case (ii) If $u \geq 0$ then

$$f(u) = \int_{0}^{\infty} e^{-(u+2v)} \, dv = e^{-u} \left(\frac{e^{-2v}}{-2} \right)_{v=0}^{v=\infty} = e^{-u} \left(0 - \frac{1}{-2} \right)$$

$$= e^{-u}/2$$

Central Limit Theorem :

If $X_1, X_2, \ldots, X_n$ be a sequence of independent identically distributed random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$, $i = 1, 2, \ldots, n$ and if $S_n = X_1 + X_2 + \cdots + X_n$, then under certain general conditions S_n follows a normal distribution with mean '$n\mu$' and variance '$n\sigma^2$' as $n \to \infty$.

Notes

* If $\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$ then $\bar{X}$ follows $N(\mu, \sigma/\sqrt{n})$ as $n \to \infty$.

Uses of Central Limit Theorem

1) It is very useful in statistical surveys for a large sample size. It helps to provide fairly accurate results.

2) It states that almost all theoritical distributions converge to normal distributions as $n \to \infty$.

3) It helps to find out the distribution of the sum of a large number of independent random variables.

4) It helps to explain the remarkable fact that the emprical frequencies of so many natural populations exhibit bell shaped (normal) curves.

Problems

① The lifetime of a certain brand of an electric bulb may be considered as a RV with mean 1200 h and S.D 250h. Find the probability, using central limit thm, that the average lifetime of 60 bulbs exceeds 1250 h.

Soln

Given $\mu = E(x) = 1200$

$\sigma = S.D \text{ of } X = 250$

$n = 60$

$\bar{X}$ = mean life of 60 bulbs.

By CLT, $\bar{X} \sim N(\mu, \sigma/\sqrt{n})$

$$\Rightarrow Z = \frac{\bar{X} - \mu}{(\sigma/\sqrt{n})}$$

Here $\sqrt{n} = \sqrt{60} = 7.75$, $\sigma/\sqrt{n} = \frac{250}{7.75} = 32.26$.

$$\therefore Z = \frac{\bar{X} - 1200}{(32.26)}$$

When $\bar{X} = 1250$, $Z = \frac{1250 - 1200}{32.26} = 1.55$

To find $P(\bar{X} > 1250)$

(i) $P(\bar{X} > 1250) = P(Z > 1.55)$

$= 0.5 - P[0 < Z < 1.55]$

$= 0.5 - 0.4394$ (using normal table)

$= 0.0606$

Unit - III

Analytic Functions.

Analytic Functions - Necessary and Sufficient conditions for Analyticity in cartesian and Polar coordinates

Analytic:- A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire function:- A function which is analytic everywhere in the finite plane is called an entire function.

Example e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ are entire functions.

Necessary condition for $f(z)$ to be analytic:-

The necessary conditions for a complex function $f(z) = u(x,y) + iv(x,y)$ to be analytic in a region R are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

(i.e) $u_x = v_y$ and $v_x = -u_y$.

C.R. eqns

$$\boxed{\begin{matrix} u_x = v_y \\ \text{and} \\ v_x = -u_y \end{matrix}}$$

Sufficient conditions for f(z) to be analytic :-

If the partial derivatives u_x, u_y, v_x and v_y are all continuous in D and $u_x = v_y$ and $u_y = -v_x$ then the function $f(z)$ is analytic in a domain D.

Polar form of C.R. eqns

If $f(z) = u(r,\theta) + iv(r,\theta)$ is differentiable at $z = re^{i\theta}$ then

$$\boxed{\begin{matrix} u_r = \frac{1}{r} v_\theta \\ v_r = -\frac{1}{r} u_\theta . \end{matrix}}$$

Problems :-

① Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Soln :-

$$f(z) = \bar{z} = \overline{(x+iy)}$$

$$= x - iy.$$

$u = x$, $v = -y$

$u_x = 1$, $v_x = 0$

$u_y = 0$, $v_y = -1$.

C.R eqns $\boxed{\begin{array}{c} u_x = v_y \\ \& \\ u_y = -v_x. \end{array}}$

Here $u_x = 1 \neq v_y$ $(\because v_y = -1)$.

$\therefore$ C.R eqns does not satisfied.

$\Rightarrow$ $f(z) = \bar{z}$ is nowhere differentiable.

(2) Show that $f(z) = |z|^2$ is differentiable at $z = 0$ but not analytic at $z = 0$.

Soln

Let $z = x + iy$.

$$f(z) = |z|^2$$
$$= z \cdot \bar{z}$$
$$= (x+iy)(x-iy)$$
$$= x^2 + y^2 + i(0)$$

$u = x^2 + y^2$, $v = 0$

$u_x = 2x$, $v_x = 0$

$u_y = 2y$, $v_y = 0$.

C.R eqns $\boxed{u_x = v_y \ \& \ v_x = -u_y}$

Here C.R eqns satisfied only at $z = 0$ and C.R eqns does not satisfy at ~~no~~ other points.

$\therefore$ $f(z) = |z|^2$ is differentiable only at $z = 0$.

Problems

1) When a function $f(z) = u+iv$ is analytic, show that $u =$ constant and $v =$ constant are orthogonal

Soln:-

Let $f(z) = u+iv$ be an analytic function

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

Given $u = a$ &

$v = b$

$u_x + u_y \frac{dy}{dx} = 0$ and

$v_x + v_y \frac{dy}{dx} = 0$.

$\Rightarrow \frac{dy}{dx} = \frac{-u_x}{u_y}$ | $\frac{dy}{dx} = \frac{-v_x}{v_y}$

$\Rightarrow m_1 = \frac{-u_x}{u_y}$ | $\Rightarrow m_2 = \frac{-v_x}{v_y}$.

$$m_1 \cdot m_2 = \left(\frac{-u_x}{u_y}\right) \cdot \left(\frac{-v_x}{v_y}\right)$$

$$= \left(\frac{-u_x}{-v_x}\right) \cdot \left(\frac{-v_x}{u_x}\right)$$

$$= -1.$$

Hence the family of curves are orthogonal.

Properties – Harmonic Conjugates :-

Formulas

1) Laplace eqn : $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$. (2 dimension)

2) Laplace operator : $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (2 dimension)

3) Laplace eqn : $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$ (3 dimension)

4) Laplace operator : $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

5) The real and imaginary parts of an analytic function are harmonic functions

6) Harmonic Function (or Potential Function)

A real function of two real variables x and y that possesses continuous second order partial derivatives and that satisfies Laplace eqn is called a harmonic function.

7) conjugate harmonic function :-

If u and v are harmonic such that $u+iv$ is analytic, then each is called a conjugate harmonic function of each other.

Problems :-

① Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

Soln :-

$$f(z) = \bar{z} = \overline{(x+iy)}$$

$$= x - iy.$$

$u = x \quad , v = -y$

$u_x = 1 \quad , v_x = 0$

$u_y = 0 \quad , v_y = -1.$

C.R eqns $\boxed{u_x = v_y \ \& \ u_y = -v_x.}$

Here $u_x = 1 \neq v_y \quad (\because v_y = -1).$

$\therefore$ C.R eqns does not satisfied.

$\Rightarrow f(z) = \bar{z}$ is nowhere differentiable.

② Show that $f(z) = |z|^2$ is differentiable at $z=0$ but not analytic at $z=0$.

Soln

Let $z = x+iy$.

$f(z) = |z|^2$

$= z.\bar{z}$

$= (x+iy)(x-iy)$

$= x^2+y^2 + i(0)$

$u = x^2+y^2$, $V = 0$

$u_x = 2x$, $V_x = 0$

$u_y = 2y$, $V_y = 0$.

C.R eqns $\boxed{u_x = V_y \ \& \ V_x = -u_y}$

Here C.R eqns satisfied only at $z=0$ and C.R eqns does not satisfy at no other points.

$\therefore f(z) = |z|^2$ is differentiable only at $z=0$.

(2) Test the analyticity of the function $f(z) = z^n$ and find its derivative.

Soln

Let $z = re^{i\theta}$

$z^n = r^n e^{in\theta}$

$= r^n \left[\cos n\theta + i\sin n\theta\right]$

$z^n = r^n \cos n\theta + i r^n \sin n\theta$

$u = r^n \cos n\theta \quad , \quad v = r^n \sin n\theta$

$\cancel{u_r = n \cdot r^{n-1} \times (-\sin n\theta \times n)}$

$u_r = n \cdot r^{n-1} \cos n\theta$

$u_\theta = -n r^n \sin n\theta$

$v_r = n \cdot r^{n-1} \sin n\theta$

$v_\theta = n r^n \cos n\theta$.

C.R. eqns in polar form.

$$\boxed{\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ v_r &= -\frac{1}{r} u_\theta \end{aligned}}$$

$u_r = n r^{n-1} \cos n\theta$

$= \frac{n \cdot r^n \cos n\theta}{r}$

$u_r = \frac{1}{r} v_\theta$

$$V_r = nr^{n-1}\sin n\theta$$

$$= \frac{nr^n \sin n\theta}{r}$$

$$= \frac{-nr^n \sin n\theta}{-r}$$

$$V_r = -\frac{1}{r} \cdot u_\theta$$

$\therefore$ C.R eqns satisfied

Also $u_r, u_\theta, V_r, V_\theta$ are continuous

$\Rightarrow$ $f(z) = z^n$ is analytic everywhere

To find $f'(z)$:-

$$f'(z) = \frac{u_r + iV_r}{e^{i\theta}}$$

$$= \frac{nr^{n-1}\cos n\theta + i n r^{n-1}\sin n\theta}{e^{i\theta}}$$

$$= \frac{nr^n \cos n\theta + i n r^n \sin n\theta}{re^{i\theta}}$$

$$= \frac{n(re^{i\theta})^n}{re^{i\theta}}$$

$$= \frac{nr^n}{z}$$

$$= nz^{n-1}$$

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Inverse Function:-

Let $w = f(z)$ be a function of z and $z = F(w)$ be its inverse function. Then the function $w = f(z)$ will cease to be analytic at points $\frac{dz}{dw} = 0$ and $z = F(w)$ will be analytic at points where $\frac{dw}{dz} = 0$.

Problems:-

① Show that the function $w = e^z$ is analytic everywhere on the complex plane.

Solution:-

Let $z = x + iy$

$$f(z) = e^z = e^{x+iy}$$

$$= e^x \cdot e^{iy}$$

$$= e^x \left[\cos y + i \sin y\right]$$

$$= e^x \cos y + i\, e^x \sin y.$$

$u = e^x \cos y$	$v = e^x \sin y$
$u_x = e^x \cos y$	$v_x = e^x \sin y$
$u_y = -e^x \sin y$	$v_y = e^x \cos y.$

$$u_x = v_y = e^x \cos y$$

&

$$v_x = -u_y = e^x \sin y.$$

C.R. eqns satisfied.

Further e^x, $\cos y$, $\sin y$ are continuous

$\Rightarrow$ u_x, u_y, v_x, v_y are continuous everywhere

$\therefore$ $f(z) = w = e^z$ is analytic everywhere.

$$f'(z) = u_x + i v_x$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x \left[\cos y + i \sin y\right]$$

$$= e^x \cdot e^{iy}$$

$$= e^{x+iy}$$

$$= e^z.$$

$\therefore$ $f'(z) = e^z.$

(3) Show that the function $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and determine its conjugate. Also find $f(z)$.

Soln:-

$$u = \frac{1}{2}\log(x^2+y^2).$$

$$u_x = \frac{x}{x^2+y^2}.$$

$$u_x(z,0) = \frac{z}{z^2} = \frac{1}{z}.$$

$$u_{xx} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Illy

$$u_y = \frac{y}{x^2+y^2}.$$

$$u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

(i) To prove u is harmonic

$$u_{xx} + u_{yy} = \frac{(y^2-x^2)+(x^2-y^2)}{(x^2+y^2)^2}$$

$$= 0.$$

$\Rightarrow u$ is harmonic.

(ii) To find $f(z)$.

$$w = f(z) = u+iv$$

$$f'(z) = u_x + iv_x$$

$$= u_x - iu_y$$

$$f(z) = \int u_x(z,0)\, dz - i\int u_y(z,0).\, dz + C.$$

$$= \int \frac{1}{z} dz - i\int 0\, dz + c$$

$$f(z). = \log z + c.$$

(iii) To find v:-

$$f(z) = \log(re^{i\theta})$$

$$u+iv = \log r + \log e^{i\theta}$$

$$= \log r + i\theta$$

$$\Rightarrow v = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

(3) Prove that $u = x^2 - y^2$, $v = -y/_{x^2+y^2}$ are harmonic functions but not harmonic conjugate.

(OR)

Give an example to show u & v are harmonic but $f(z) = u + iv$ is not analytic.

Soln.

$u = x^2 - y^2$

$u_x = 2x$, $u_{xx} = 2$. $\Rightarrow u_{xx} + u_{yy} = 2 - 2 = 0$

$u_y = -2y$, $u_{yy} = -2$.

$\therefore$ u is harmonic.

$v = -y/_{x^2+y^2}$

$$v_x = \frac{(x^2+y^2)(0) - [(-y)(2x)]}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}.$$

$$v_{xx} = \frac{(x^2+y^2)^2\, 2y - 2xy[2(x^2+y^2) \cdot 2x]}{(x^2+y^2)^4}$$

$$= \frac{2(x^2+y^2)^2 y - 8x^2y\,(x^2+y^2)}{(x^2+y^2)^4}.$$

$$v_y = \frac{(x^2+y^2)(-1) - [-y\,(2y)]}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

Problem 1 Prove that the real and imaginary parts of an analytic function are harmonic function.

Soln

Let $f(z) = u + iv$ be analytic.

$\Rightarrow$ C-R eqn satisfied

$$\Rightarrow \quad u_x = v_y \longrightarrow (1)$$

&

$$u_y = -v_x \longrightarrow (2).$$

Diff (1) & (2) partially w.r.t 'x'.

$$u_{xx} = v_{xy} \longrightarrow (3)$$

&

$$u_{xy} = -v_{xx}. \longrightarrow (4)$$

Diff (1) & (2) partially w.r.t 'y'.

$$u_{yx} = v_{yy} \longrightarrow (5)$$

$$u_{yy} = -v_{yx}. \longrightarrow (6).$$

$$(3) + (6) \Rightarrow u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0$$

$$(5) - 4 \Rightarrow v_{yy} + v_{xx} = u_{yx} - u_{xy} = 0.$$

$\therefore$ u and v are harmonic.

(2) If $f(z) = u+iv$ is a regular function of z in a domain D then $\nabla^2 |f(z)|^2 = 4|f'(z)|^2$.

Soln.

$$f(z) = u+iv$$

$$|f(z)| = \sqrt{u^2+v^2}.$$

$$|f(z)|^2 = u^2+v^2.$$

$$f'(z) = u_x + iv_x.$$

$$|f'(z)| = \sqrt{(u_x)^2+(v_x)^2}$$

$$|f'(z)|^2 = (u_x)^2+(v_x)^2$$

$$\nabla^2 |f(z)|^2 = \nabla^2 (u^2+v^2)$$

$$= \nabla^2 (u^2) + \nabla^2 (v^2).$$

$$\nabla^2 (u^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u^2$$

$$= \frac{\partial^2 (u^2)}{\partial x^2} + \frac{\partial^2 (u^2)}{\partial y^2}.$$

$$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left[2u \cdot \frac{\partial u}{\partial x}\right]$$

$$= 2\left[u \cdot \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial x}\right].$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2u \cdot \frac{\partial^2 u}{\partial x^2} + 2 \cdot \left(\frac{\partial u}{\partial x}\right)^2$$

|||ly $$\frac{\partial^2}{\partial y^2}(u^2) = 2u \cdot \frac{\partial^2 u}{\partial y^2} + 2 \cdot \left(\frac{\partial u}{\partial y}\right)^2.$$

$$\therefore \nabla^2(u^2) = 2u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 0 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \quad (\because u \text{ is harmonic})$$

$$\nabla^2(u^2) = 2u_x^2 + 2u_y^2$$

$\|^{ly}$ $\nabla^2(v^2) = 2v_x^2 + 2v_y^2.$

$$\therefore \nabla^2|f(z)|^2 = 2\left[u_x^2 + u_y^2 + v_x^2 + v_y^2\right]$$

$$= 2\left[u_x^2 + (-v_x)^2 + v_x^2 + u_x^2\right]$$

$$= 4\left[u_x^2 + v_x^2\right]$$

$$\nabla^2|f(z)|^2 = 4\cdot|f'(z)|^2.$$

3) If $f(z) = u+iv$ is a regular function of z in a domain D, then

$\nabla^2 \log|f(z)| = 0$ if $f(z)\cdot f'(z) \neq 0$ in D.

(or) $\log|f(z)|$ is harmonic in D.

Soln

$f(z) = u+iv.$

$|f(z)| = \sqrt{u^2+v^2}.$

$$\log|f(z)| = \frac{1}{2}\log(u^2+v^2)$$

$$\nabla^2\log|f(z)| = \frac{1}{2}\nabla^2\log(u^2+v^2)$$

$$= \frac{1}{2}\left[\frac{\partial^2}{\partial x^2}\log(u^2+v^2) + \frac{\partial^2}{\partial y^2}\log(u^2+v^2)\right]$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\log(u^2+v^2)\right] = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{1}{u^2+v^2}\left[2u\cdot\frac{\partial u}{\partial x} + 2v\cdot\frac{\partial v}{\partial x}\right]\right]$$

$$= \frac{1}{2}\times 2\cdot\frac{\partial}{\partial x}\left[\frac{u\cdot u_x + vv_x}{u^2+v^2}\right]$$

$$= \frac{(u^2+v^2)\left[u\cdot u_{xx} + u_x\cdot u_x + v\cdot v_{xx} + v_xv_x\right] - \left[u\cdot u_x + vv_x\right]\left[2u\cdot u_x + 2v\cdot v_x\right]}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)\left[uu_{xx} + vv_{xx} + u_x^2 + v_x^2\right] - 2(uu_x + vv_x)^2}{(u^2+v^2)^2}$$

||ly $\frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\log(u^2+v^2)\right]$

$$= \frac{(u^2+v^2)\left[u\cdot u_{yy} + v\cdot v_{yy} + u_y^2 + v_y^2\right] - 2(uu_y + vv_y)^2}{(u^2+v^2)^2}$$

$$\therefore \nabla^2 \log|f(z)| = \frac{-2\left[uu_x + vv_x\right]^2 - 2\left[uu_y + vv_y\right]^2}{(u^2+v^2)^2}$$

$$= \frac{2(u^2+v^2)|f'(z)|^2 - 2(u^2+v^2)|f'(z)|^2}{(u^2+v^2)^2}$$

$$\nabla^2 \log|f(z)| = 0.$$

(4) If $f(z) = u+iv$ is a regular function of z in a domain D, then

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2.$$

Soln:-

$$f(z) = u+iv$$

$$|f(z)| = \sqrt{u^2+v^2}.$$

$$\frac{\partial}{\partial x}|f(z)| = \frac{\partial}{\partial x}(u^2+v^2)^{1/2}$$

$$= \frac{1}{2(u^2+v^2)^{1/2}}\left[2u.u_x + 2v.v_x\right]$$

$$= \frac{uu_x + v.v_x}{\sqrt{u^2+v^2}}.$$

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$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 = \frac{u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x}{u^2+v^2}$$

$$|||^{ly} \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = \frac{u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y}{u^2+v^2}$$

$$\therefore \left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2$$

$$= \frac{u^2\left[u_x^2 + u_y^2\right] + v^2\left[v_x^2 + v_y^2\right] + 2uv\left[u_xv_x + u_yv_y\right]}{u^2+v^2}$$

$$= \frac{u^2|f'(z)|^2 + v^2|f'(z)|^2 + 2uv(0)}{u^2+v^2}$$

$$= |f'(z)|^2$$

L.H.S = R.H.S.

(5) Find the value of m if $u = 2x^2 - my^2 + 3x$ is harmonic.

Soln $u = 2x^2 - my^2 + 3x$

Harmonic : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

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$$\frac{\partial u}{\partial x} = 4x + 3.$$

$$\frac{\partial^2 u}{\partial x^2} = 4.$$

$$\frac{\partial u}{\partial y} = -2my$$

$$\frac{\partial^2 u}{\partial y^2} = -2m.$$

$$\Rightarrow \quad 4 - 2m = 0.$$

$$\Rightarrow \quad 4 = 2m$$

$$\boxed{m = 2}$$

Construction of Analytic Function:-

Method(1):- (i) Suppose the harmonic function $u(x,y)$ is given.

$dv = V_x\, dx + V_y\, dy$ is an exact differential where V_x and V_y are known from u by using C.R eqns.

$$\therefore V = \int V_x\, dx + \int V_y\, dy = -\int u_y\, dx + \int u_x\, dy.$$

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(ii) Suppose the harmonic function $v(x,y)$ is given

$$du = u_x\,dx + u_y\,dy$$ is an exact differential

where u_x and u_y are known from v by using C.R eqns.

$$u = \int u_x\,dx + \int u_y\,dy$$

$$= \int v_y\,dx + \int -v_x\,dy.$$

$$= \int v_y\,dx - \int v_x\,dy.$$

Method (2) :- Substitution Method.

$$f(z) = 2u\left[\frac{1}{2}(z+a),\ \frac{-i}{2}(z-a)\right] - \left[u(a,0) - iv(a,0)\right].$$

Here $u(a,0) - iv(a,0)$ is a constant.

$$\therefore\ f(z) = 2u\left[\frac{1}{2}(z+a),\ \frac{-i}{2}(z-a)\right] + C.$$

By taking $a=0$

$$f(z) = 2\left[u\left(\frac{z}{2},\ \frac{-iz}{2}\right)\right] + C.$$

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Method(3) :- [Milne – Thomson Method].

(i) To find $f(z)$ when u is given.

$$f(z) = u + iv.$$

$$f'(z) = u_x + iv_x.$$

$$= u_x - iu_y \quad [\text{by C.R. eqn}]$$

$$\therefore f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + C.$$

where $C \to$ complex constant.

(ii) To find $f(z)$ when v is given

$$f(z) = u + iv$$

$$f'(z) = u_x + iv_x$$

$$= v_y + iv_x.$$

$$\therefore f(z) = \int v_y(z,0)\,dz + i\int v_x(z,0)\,dz + C.$$

where $C \to$ complex constant.

Problems

① Find a function w such that $w = u + iv$ is analytic if $u = e^x \sin y$.

Soln:-

$$u = e^x \sin y$$

$$u_x = e^x \sin y$$

$$u_x(z,0) = 0$$

$$u_y = e^x \cos y$$

$$u_y(z,0) = e^z$$

$$w = f(z) = u + iv$$

$$f'(z) = u_x + iv_x$$

$$= u_x - iu_y \quad \text{(by C.R eqn)}$$

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + C$$

where C is a complex constant.

$$\therefore f(z) = \int 0\,dz - i\int e^z\,dz + C$$

$$= -ie^z + C.$$

(2) Determine the analytic function $w = u + iv$

if $u = e^{2x}(x\cos 2y - y\sin 2y)$.

<u>Soln</u>

$$u = e^{2x}\left[x\cos 2y - y\sin 2y\right].$$

$$u_x = e^{2x}\cos 2y + \left[x\cos 2y - y\sin 2y\right] 2.e^{2x}$$

$$u_x(z,0) = e^{2z} + \left[z(1) - 0\right].\ 2e^{2z}.$$

$$= e^{2z} + 2ze^{2z}.$$

$$= (1+2z)e^{2z}.$$

$$u_y = e^{2x}\left[-2x\sin 2y - (2y\cos 2y + \sin 2y)\right]$$

$$= e^{2z}\left[-0 - (0+0)\right]$$

$$= 0.$$

Let $w = f(z) = u + iv$

$$f'(z) = u_x + iv_x$$

$$= u_x - iu_y$$

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + C.$$

$$= \int (1+2z).e^{2z}\,dz - i\int 0\,dz + c.$$

$$f(z) = \int (1+2z).\, e^{2z}\, dz + c$$

$$= (1+2z)\frac{e^{2z}}{2} - (2).\frac{e^{2z}}{4} + c$$

$$= \frac{e^{2z}}{2} + z.e^{2z} - \frac{e^{2z}}{2} + c$$

$$= z.e^{2z} + c.$$

(3) Determine the analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

Soln

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}.$$

$$u_x = \frac{[\cosh 2y - \cos 2x][2\cos 2x] - \sin 2x\,[2\sin 2x]}{(\cosh 2y - \cos 2x)^2}$$

$$u_x(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{(1-\cos 2z)^2}.$$

$$= \frac{-2}{1-\cos 2z}$$

$$= \frac{-2}{2\sin^2 z}.$$

$$u_x(z,0). = -\operatorname{cosec}^2 z.$$

$$u_y = \frac{(\cosh ay - \cos ax)(0) - \sin ax\,[a \sinh ay]}{(\cosh ay - \cos ax)^2}.$$

$$\boxed{u_y(z,0) = 0.}$$

$$w = f(z)$$

$$= u + iv$$

$$f'(z) = u_x + iv_x$$

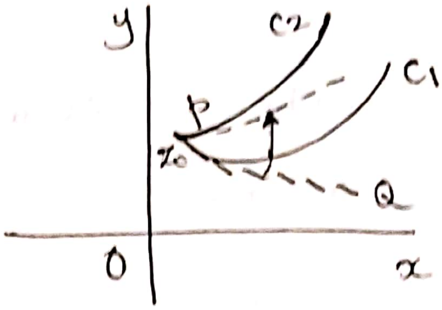
$$= u_x - iu_y$$

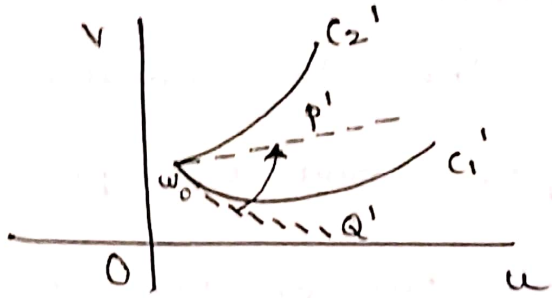
$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + C$$

$$f(z) = \int (-\operatorname{cosec}^2 z)\,dz - i\int 0\,dz + C$$

$$\boxed{f(z) = \cot z + C.}$$

Conformal Mappings, $w = z + k$, $w = kz$, $w = \frac{1}{z}$, $w = z^2$.





If the angle between C_1 and C_2 at z_0 is the same as the angle between C_1' & C_2' at w_0 both in magnitude and sence, then the transformation $w = f(z)$ is said to be conformal at z_0

Isogonal :- A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sence is said to be an isogonal at the point

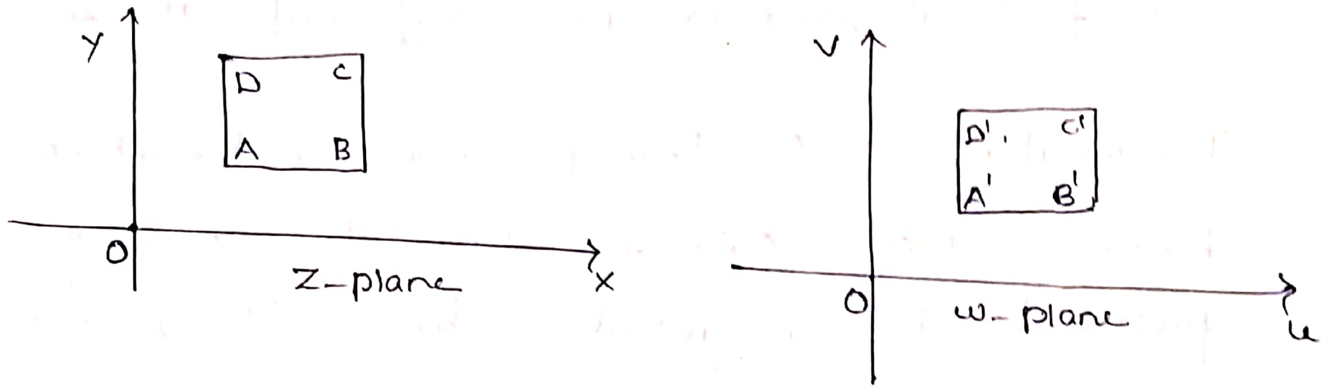
* A mapping $w = f(z)$ is said to be conformal at $z = z_0$ if $f'(z_0) \neq 0$.

* The point at which the mapping $w = f(z)$ is not conformal (ie) $f'(z) = 0$ is called a critical point of the mapping.

1) Translation:-

The transformation $w = C + z$, where C is a complex constant, represents the relation.

This transformation transforms a circle into an equal circle. Also the corresponding regions in the z and w planes will have the same shape, size and orientation.



2) Magnification:-

The transformation $w = cz$, where c is a real constant, represents magnification.

The size of any figure in the z-plane is magnified c times, but there will be no change in the shape and orientation. This transformation transforms circles into circles.

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3) Magnification and Rotation:-

The transformation $w = cz$ where c is a complex constant, represents both magnification and rotation.

4) Magnification, Rotation and Translation:-

The linear transformation $w = az + b$ where a and b are complex constants, represents magnification, rotation and translation.

Under this transformation circles will be mapped into circles.

5) Inversion and Reflection:-

The transformation $w = \frac{1}{z}$ represents inversion w.r.t the unit circle $|z| = 1$ followed by reflection in the real axis.

Under the transformation $w = \frac{1}{z}$ a circle in z plane transform to another circle in the w-plane. When circle pass through the origin it will have the image as a straight line under this transformation.

Problems :

① Find the image of the circle $|z| = \lambda$ under the transformation $w = 5z$.

Soln

$$w = 5z.$$

$$|w| = 5|z|$$

$$|w| = 5\lambda.$$

Hence the image of $|z| = \lambda$ in the z-plane is transformed into $|w| = 5\lambda$ in the w-plane under the transformation $w = 5z$.

② Find the image of the region $y > 1$ under the transformation $w = (1-i)z$.

Soln

$$w = (1-i)z.$$

$$u + iv = (1-i)(x+iy)$$

$$= (x+y) + i(y-x).$$

$$u = x+y, \quad v = y-x.$$

$$u+v = 2y, \quad u-v = 2x$$

$$y = \frac{u+v}{2}, \quad x = \frac{u-v}{2}.$$

Hence image region $y > 1$ is $\frac{u+v}{2} > 1$.

(i.e) $u + v > 2$ in w-plane.

Conformal Mapping

Problems Based On $\omega = \frac{1}{z}$.

$$\omega = u + iv$$
$$z = x + iy.$$

Problems

① Find the image of $|z - 2i| = 2$ under the transformation $\omega = 1/z$.

Soln. Given $|z - 2i| = 2$.

$$\omega = 1/z \Rightarrow z = \frac{1}{\omega}.$$

$$\therefore \left| \frac{1}{\omega} - 2i \right| = 2$$

$$\left| \frac{1 - 2\omega i}{\omega} \right| = 2$$

$$\Rightarrow |1 - 2\omega i| = 2|\omega|$$

$$\Rightarrow |1 - 2i(u + iv)| = 2|u + iv|$$

$$\Rightarrow |1 - 2iu - 2i^2 v| = 2|u + iv|$$

$$\Rightarrow |(1 + 2v) + i(-2u)| = 2|u + iv|$$

$$\Rightarrow \sqrt{(1 + 2v)^2 + (-2u)^2} = 2\sqrt{u^2 + v^2}$$

$\Rightarrow$ Squaring both sides we get

$$(1 + 2v)^2 + (-2u)^2 = 4(u^2 + v^2)$$

$$1 + 4v + 4v^2 + 4u^2 = 4u^2 + 4v^2$$

$$\Rightarrow 1 + 4v + 4v^2 + 4u^2 - 4u^2 - 4v^2 = 0$$

$$\Rightarrow 1 + 4v = 0 \Rightarrow \boxed{v = -1/4}$$ is required image.

② Find the image of the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ under the transformation $w = 1/z$.

Soln

$$w = 1/z$$

$$\Rightarrow z = 1/w$$

$$\Rightarrow x + iy = \frac{1}{u+iv}$$

$$= \frac{u-iv}{(u+iv)(u-iv)}$$

$$= \frac{u-iv}{u^2-(iv)^2}$$

$$x+iy = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} + i\frac{(-v)}{u^2+v^2}$$

Equating real and imaginary parts both sides

$$\boxed{x = \frac{u}{u^2+v^2}}, \quad \boxed{y = \frac{-v}{u^2+v^2}}$$

Given infinite strip : $\frac{1}{4} < y < \frac{1}{2}$.

First we find the image of $y = 1/4$

Second we find the image of $y = 1/2$.

(i) To find image of $y = 1/4$

W.K.T $y = \frac{-v}{u^2+v^2}$

$$\Rightarrow y = \frac{-v}{u^2+v^2} = 1/4$$

$$\Rightarrow -4v = u^2+v^2$$

$$\Rightarrow \boxed{u^2+v^2+4v = 0}$$ which is a circle

(3) Find the image of $|z-2i| = 2$ under the transformation $w = \frac{1}{z}$.

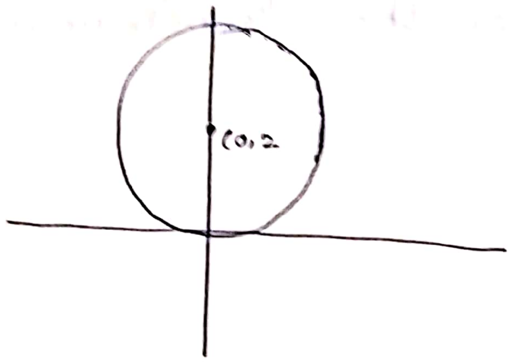
Soln

$|z-2i| = 2$ is a circle.

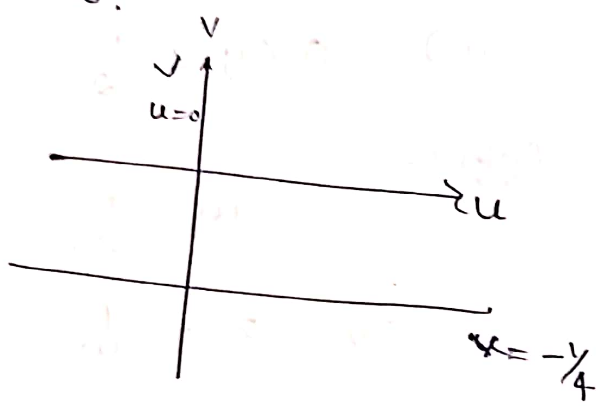
centre $= 2i = (0, 2)$

radius $= 2$.

$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$.



z plane



w-plane

(1) $\Rightarrow \left|\frac{1}{w} - 2i\right| = 2.$

$\Rightarrow |1 - 2wi| = 2|w|$

$\Rightarrow |1 - 2(u+iv)i| = 2|u+iv|$

$\Rightarrow |(1+2v) - i(2u)| = 2|u+iv|$

$= \sqrt{(1+2v)^2 + (2u)^2} = 2\sqrt{u^2+v^2}$.

$\Rightarrow (1+2v)^2 + (2u)^2 = 4(u^2+v^2)$.

$$1+4v^2+4v+4u^2 = 4u^2+4v^2$$

$$1+4v = 0.$$

$v = -\frac{1}{4}$ which is a straight line in ω-plane

④ Find the image of the infinite strips

(i) $\frac{1}{4} < y < \frac{1}{2}$

(ii) $0 < y < \frac{1}{2}$ under the transformation $\omega = \frac{1}{z}$

Soln :-

$$\omega = \frac{1}{z}.$$

$$\Rightarrow \quad z = \frac{1}{\omega}.$$

$$z = \frac{1}{(u+iv)}$$

$$= \frac{u-iv}{(u-iv)(u+iv)}$$

$$= \frac{u-iv}{u^2+v^2}.$$

$$x+iy = \frac{u-iv}{u^2+v^2}.$$

$$= \left(\frac{u}{u^2+v^2}\right) + i\left(\frac{-v}{u^2+v^2}\right).$$

$x = \frac{u}{u^2+v^2} \longrightarrow ①$, $y = \frac{-v}{u^2+v^2} \longrightarrow ②$.

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(i) Given strip $\frac{1}{4} < y < \frac{1}{2}$.

When $y = \frac{1}{4}$.

$$\frac{1}{4} = \frac{-v}{u^2+v^2} \Rightarrow \begin{aligned} u^2+v^2 &= -4v \\ u^2+(v^2+4v) &= 0 \\ u^2+(v^2+4v+4) &= 4. \end{aligned}$$

$$u^2 + (v+2)^2 = 4 \longrightarrow (3) \quad (\because \text{adding } (\text{coeff } v/2)^2 = (4/2)^2)$$

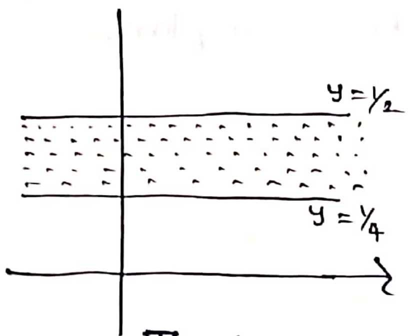
which is a circle whose centre is $(0,-2)$ in the w-plane and radius is 2.

When $y = \frac{1}{2}$

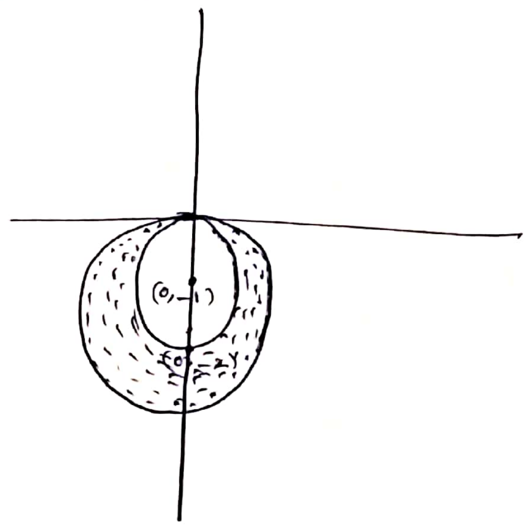
$$\frac{1}{2} = \frac{-v}{u^2+v^2} \Rightarrow \begin{aligned} u^2+v^2 &= -2v. \\ u^2+v^2+2v &= 0. \\ u^2+(v^2+2v+1) &= 1. \end{aligned}$$

$$u^2+v^2+2v = 0.$$

$u^2 + (v+1)^2 = 1$. which is a circle whose centre is $(0,-1)$ in the w-plane and unit radius.



Z-plane



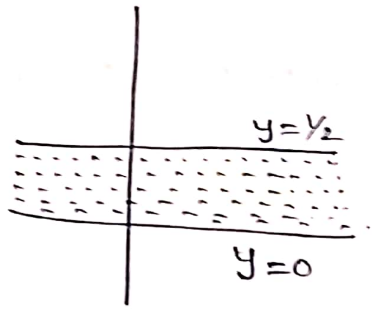
w-plane.

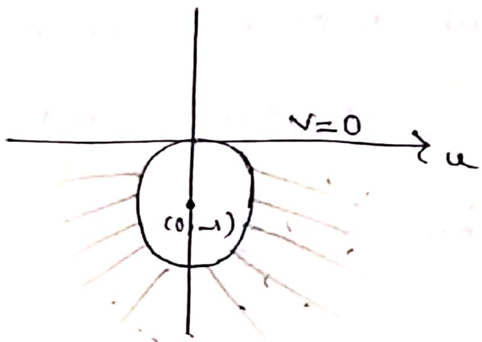
Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region between the circles

$u^2+(v+1)^2 = 1$ &

$u^2+(v+2)^2 = 4$ in w-plane.

(ii) Given strip is $0 < y < \frac{1}{2}$.





When $y=0$

$\Rightarrow v=0$.

When $y=\frac{1}{2}$ we get $u^2+(v+1)^2 = 1$

Hence the infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2+(v+1)^2=1$ in the lower half of the w-plane.

(5) Find the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$.

Soln

$$w = \frac{1}{z}$$

$$\Rightarrow \quad z = \frac{1}{w}.$$

$$\Rightarrow \quad x + iy = \frac{1}{Re^{i\phi}}$$

$$= \frac{1}{R} e^{-i\phi}$$

$$= \frac{1}{R}\left[\cos\phi - i\sin\phi\right]$$

$$x = \frac{1}{R}\cos\phi, \quad y = -\frac{1}{R}\sin\phi.$$

Given $x^2 - y^2 = 1.$

$$\Rightarrow \quad \left[\frac{1}{R}\cos\phi\right]^2 - \left[-\frac{1}{R}\sin\phi\right]^2 = 1.$$

$$\frac{\cos^2\phi - \sin^2\phi}{R^2} = 1.$$

$$\cos 2\phi = R^2$$ which is lemniscate.

⑥ Find the critical points of the transformation

$$w^2 = (z-\alpha)(z-\beta)$$

Soln :-

$$w^2 = (z-\alpha)(z-\beta).$$

Critical points occur at $\frac{dw}{dz} = 0$ and

$\frac{dz}{dw} = 0.$

$$2w \cdot \frac{dw}{dz} = (z-\alpha) + (z-\beta)$$

$$\frac{dw}{dz} = \frac{2z - (\alpha+\beta)}{2w}.$$

Case (i) $\frac{dw}{dz} = 0$

$$\Rightarrow \frac{2z - (\alpha+\beta)}{2w} = 0$$

$$\Rightarrow \boxed{z = \frac{\alpha+\beta}{2}}$$

Case (ii) $\frac{dz}{dw} = 0$

$$\Rightarrow \frac{2w}{2z-(\alpha+\beta)} = 0$$

$$\Rightarrow \frac{w}{z - \left(\frac{\alpha+\beta}{2}\right)} = 0.$$

$\Rightarrow w = 0 \Rightarrow (z-\alpha)(z-\beta) = 0.$

$z = \alpha, \beta.$

$\therefore$ The critical points are $\alpha, \beta, \frac{\alpha+\beta}{2}.$

Bilinear Transformation

The transformation $w = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ is called a bilinear transformation.

Here a, b, c, d are complex numbers.

This transformation is also called by Mobius transformation.

* Under a bilinear transformation no two points in z-plane go to the same point in w-plane.

* The bilinear transformation has atmost two fixed points.

* The bilinear transformation which transforms z_1, z_2, z_3 into w_1, w_2, w_3 is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

CROSS RATIO:-

Given four points z_1, z_2, z_3, z_4 in this order, the ratio $\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called cross ratio of the points.

* Bilinear Transformation is conformal only when $\frac{dw}{dz} \neq 0$

* The inverse of the transformation $w = \frac{az+b}{cz+d}$ is $z = \frac{-dw+b}{cw-a}$ which is also a bilinear transformation except $w = \frac{a}{c}$.

* Each point in the plane except $z = -\frac{d}{c}$ corresponds to a unique point in the w-plane. The point $z = -\frac{d}{c}$ corresponds to the point at infinity in the w-plane.

* The cross ratio of four points

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

is invarient under bilinear transformation.

* Suppose $z_1 = \infty$ then $w = \frac{z - z_1}{z_2 - z_1}$.

Problems

① Find the invariant point of the bilinear transformation $w = \frac{1+z}{1-z}$.

Soln

The invariant points are given by (Put $w=z$)

$$z = \frac{1+z}{1-z}.$$

$$z - z^2 = 1 + z.$$

$$z^2 = -1.$$

$$z = \pm i.$$

$\therefore$ i and $-i$ are invariant points.

② Find the fixed points of the transformation

$$w = \frac{2z+6}{z+7}.$$

Soln:-

Put $w = z$.

$$z = \frac{2z+6}{z+7}.$$

$$z^2 + 7z = 2z + 6.$$

$$z^2 + 5z - 6 = 0.$$

$$z = -6, \; z = 1.$$

(3) Find the fixed points under the transformation

$$w = \frac{2z-5}{z+4}.$$

Soln:-

Put $w = z$.

$$z = \frac{2z-5}{z+4}.$$

$$z^2 + 4z = 2z - 5.$$

$$z^2 + 2z + 5 = 0.$$

$$z = \frac{-2 \pm \sqrt{4-20}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm i4}{2}$$

$$= -1 \pm i2.$$

$\therefore$ Fixed points are $-1+i2$ & $-1-i2$

(4) Find the bilinear transformation that maps the points $z = 0, -1, i$ into the points $w = i, 0, \infty$ respectively.

Soln:-

$z_1 = 0, \quad z_2 = -1, \quad z_3 = i$

$w_1 = i, \quad w_2 = 0, \quad w_3 = \infty.$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Omit the factors involving w_3 ($\because w_3 = \infty$)

$$\frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w-i}{0-i} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$w-i = \frac{z}{z-i}(-i+1).$$

$$w = \frac{z}{z-i}(-i+1)+i$$

$$= \frac{-iz+z+iz+1}{(z-i)}$$

$$w = \frac{z+1}{(z-i)}$$

(5) Find the bilinear transformation which maps the points $1, i, -1$ onto the points $0, 1, \infty$, show that the transformation maps the interior point of the unit circle of the z-plane onto the upper half of the w-plane.

Soln

Given $z_1 = 1$, $z_2 = i$, $z_3 = -1$

$w_1 = 0$, $w_2 = 1$, $w_3 = \infty$.

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w-0}{1-0} = \frac{(z-1)(i+1)}{(z+1)(i-1)}.$$

$$w = \frac{(z-1)(i+1)}{(z+1)(i-1)}.$$

$$w = \frac{(z-1)}{(z+1)}(-i)$$

$$= \frac{(-i)(z)+i}{1(z)+1}.$$

To find z.

$$wz + w = -iz + i.$$

$$wz + iz = -w + i$$

$$z\left[w+i\right] = -(w) + i$$

$$z = \frac{-(w-i)}{w+i}.$$

To prove $|z| < 1$ maps $v > 0$.

$$|z| < 1$$

$$\Rightarrow \left|\frac{-(w-i)}{w+i}\right| < 1.$$

$$\Rightarrow |w-i| < |w+i|$$

$$\Rightarrow |u+iv-i| < |u+iv+i|$$

$$\Rightarrow u^2 + (v-1)^2 < u^2 + (v+1)^2$$

$$\Rightarrow -4v < 0$$

$$\Rightarrow v > 0.$$

H/P.

Bilinear Transformation

z plane : z_1, z_2, z_3
w plane : w_1, w_2, w_3 } Given

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Problem Find the bilinear transformation which maps points $z: 0, 1, -1$ onto the points $w: -1, 0, \infty$. Also find the invariant (fixed) points.

Soln: Given $z_1 = 0$, $z_2 = 1$, $z_3 = -1$

$w_1 = -1$, $w_2 = 0$, $w_3 = \infty$.

Formula $$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Since $w_3 = \infty$, formula becomes

$$\frac{(w-w_1)}{(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w-(-1)}{0-(-1)} = \frac{(z-0)(1-(-1))}{(z-(-1))(1-0)}$$

$$\frac{w+1}{1} = \frac{z(1+1)}{(z+1)(1)}$$

$$w+1 = \frac{2z}{(z+1)}$$

$$w = \frac{2z}{(z+1)} - 1$$

$$w = \frac{2z-(z+1)}{(z+1)} = \frac{2z-z-1}{(z+1)}$$

$$\boxed{w = \frac{z-1}{z+1}}$$ is required bilinear transformation.

To find fixed points

$$w = \frac{z-1}{z+1}.$$

Put $w = z$

$$z = \frac{z-1}{z+1}$$

$$z(z+1) = z-1.$$

$$z^2+z = z-1.$$

$$z^2+z-z+1 = 0.$$

$$z^2+1 = 0.$$

$$z = \pm i.$$

$\therefore$ Fixed points are $i, -i$.

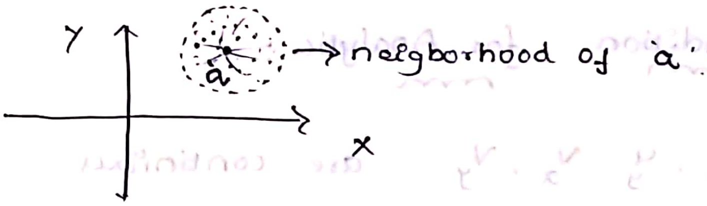
$f(z) = u + iv$

$u \rightarrow$ real part of $f(z)$

$v \rightarrow$ imaginary part of $f(z)$.

Analytic Function

A function $f(z)$ is analytic at a point 'a' if its derivative exists at some neighborhood of 'a'.



Entire Function:

A function $f(z)$ is analytic at every point in a complex plane then it is entire function.

Examples e^z, $\sin z$, $\cos z$ are entire functions.

Notation

$$\frac{\partial u}{\partial x} = u_x, \quad \frac{\partial u}{\partial y} = u_y$$

$$\frac{\partial v}{\partial x} = v_x, \quad \frac{\partial v}{\partial y} = v_y.$$

Necessary condition for Analytic

If $f(z) = u + iv$ is analytic then

$$\boxed{\begin{array}{c} u_x = v_y \\ \& \\ u_y = -v_x \end{array}} \rightarrow \text{Cauchy Riemann eqn } (\text{C-R eqn}).$$

Sufficient condition for Analytic :

(i) u_x, u_y, v_x, v_y are continious

(ii) $$\boxed{\begin{array}{c} u_x = v_y \\ \& \\ u_y = -v_x \end{array}}$$

The above are sufficient conditions for $f(z) = u + iv$ is analytic.

Polar form of C-R eqn :

$$\boxed{\begin{array}{c} u_r = \frac{1}{r} v_\theta \\ \\ v_r = -\frac{1}{r} u_\theta \end{array}}$$

Milne - Thomson Method

We use this method to find $f(z)$ when the real part of $f(z)$ is given

(or)

the imaginary part of $f(z)$ is given.

Formula ①: When the real part of $f(z) = u$ is given

$$\boxed{f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + C}$$

Formula ②: When the imaginary part of $f(z) = V$ is given

$$\boxed{f(z) = \int V_y(z,0)\,dz + i\int V_x(z,0)\,dz + C}$$

Note

* $\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$

Here $u', u'', u''', \dots$ are successive differentiations of u

$V_1, V_2, V_3, \dots$ are successive integrations of V.

Problem ① Find the analytic function whose real part is given by $e^x \cos y$.

Soln. Given $u = e^x \cos y$.

$$u_x = e^x \cos y.$$

$$u_x(z,0) = e^z \cos 0 = e^z$$

$$u_y = -e^x \sin y$$

$$u_y(z,0) = -e^z \sin 0 = 0.$$

Formula ①

$$f(z) = \int u_x(z,0)\, dz - i \int u_y(z,0)\, dz + C$$

$$= \int e^z\, dz - i \int 0\, dz + c$$

$$= \int e^z\, dz + c$$

$$= e^z + c.$$

② Determine the analytic function whose real part is $y + e^x \cos y$.

Soln. Given $u = y + e^x \cos y$

$$u_x = e^x \cos y$$

$$u_x(z,0) = e^z \cos 0 = e^z$$

$$u_y = 1 - e^x \sin y$$

$$u_y(z,0) = 1 - e^z \sin 0 = 1.$$

Formula ①

$$f(z) = \int u_x(z,0)\, dz - i \int u_y(z,0)\, dz + C$$

$$= \int e^z\, dz - i \int 1 .\, dz + C$$

$$= e^z - iz + C$$

③ Determine the analytic function whose real part is $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Soln.

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

$$u_x = 3x^2 - 3y^2 + 6x$$

$$u_x(z,0) = 3z^2 - 3(0)^2 + 6z$$

$$= 3z^2 + 6z.$$

$$u_y = -3x(2y) - 3(2y)$$

$$= -6xy - 6y$$

$$u_y(z,0) = -6z(0) - 6(0) = 0$$

Formula ①

$$f(z) = \int u_x(z,0)\, dz - i \int u_y(z,0)\, dz + C$$

$$= \int (3z^2 + 6z)\, dz - i \int 0\, dz + C$$

$$= \int (3z^2 + 6z)\, dz + C$$

$$= 3z^3/3 + 6z^2/2 + C$$

$$= z^3 + 3z^2 + C.$$

Conjugate Harmonic Function

$f(z) = u + iv$ is analytic

Then harmonic conjugate of u is 'v'

harmonic conjugate of v is '$-u$'

(4) Find an analytic Function whose real part is given by $e^x(x\cos y - y\sin y)$. Also find the conjugate harmonic function of 'u'.

Soln:

Given $u = e^x(x\cos y - y\sin y)$

$$u = e^x \cdot x\cos y - e^x y\sin y$$

$$u_x = \cos y\left[e^x(1) + x \cdot e^x\right] - e^x y\sin y$$

$$u_x(z,0) = \cos 0\left[e^z + ze^z\right] - e^z(0)\sin 0$$

$$\boxed{u_x(z,0) = e^z + ze^z}$$

$$u_y = -e^x \cdot x\sin y - e^x\left[y\cos y + \sin y(1)\right]$$

$$= -e^x x\sin y - e^x \cdot y\cos y - e^x\sin y$$

$$u_y(z,0) = -e^z z.\sin(0) - e^z(0)\sin 0 - e^z\sin 0$$

$$\boxed{u_y(z,0) = 0}$$

Formula ①

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + C$$

$$= \int (e^z + ze^z)\,dz - i\int 0\,dz + C$$

$$= \int e^z\,dz + \int ze^z\,dz + C.$$

$$= e^z + \left[z.e^z - 1e^z + 0\right] + C$$

$$= e^z + ze^z - e^z + C$$

$$\boxed{f(z) = ze^z + C}$$ is required analytic function.

$\int ze^z dz$

Take

$u = z \qquad dv = e^z dz$

$v = \int dv$

$u' = 1 \qquad v = \int e^z dz = e^z$

$u'' = 0 \qquad v_1 = \int e^z dz = e^z$

$v_2 = e^z$

$\int u\,dv = uv - u'v_1 + u''v_2 - \dots$

To find harmonic conjugate of u

$$f(z) = ze^z + c.$$

Put $z = x+iy$. in R.H.S

$$f(z) = (x+iy)\,e^{(x+iy)} + c$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$= (x+iy)\,e^x . e^{iy} + c$$

$$= (x+iy)\,e^x . (\cos y + i\sin y) + c$$

$$= (x+iy).(e^x\cos y + ie^x\sin y) + c$$

$$= (xe^x\cos y - e^x y\sin y) + ie^x(y\cos y + x\sin y)$$

$$= u + iv$$

where $v = e^x(y\cos y + x\sin y)$ is harmonic conjugate of 'u'.

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⑤ Determine the analytic function whose real part is $\dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

Soln. Given: $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x = \frac{\left[(\cosh 2y - \cos 2x)\, 2\cos 2x\right] - \left[\sin 2x\,(0 + 2\sin 2x)\right]}{(\cosh 2y - \cos 2x)^2}$$

$$u_x(z,0) = \frac{(\cosh 2(0) - \cos 2z)\, 2\cos 2z - 2\sin^2 2z}{\left[\cosh 2(0) - \cos 2z\right]^2}$$

$\boxed{\cosh 0 = 1}$

$$= \frac{(1 - \cos 2z)\, 2\cos 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} = \frac{-2(1 - \cos 2z)}{(1 - \cos 2z)^2}$$

$$= -2/(1 - \cos 2z)$$

$\boxed{1 - \cos 2z = 2\sin^2 z}$

$$= -2/2\sin^2 z$$

$$= -1/\sin^2 z = -\operatorname{cosec}^2 z.$$

Formula ②

$$f(z) = \int V_y(z,0)\,dz + i\int V_x(z,0)\,dz + C$$

$$= \int 0\,dz + i\int (-ze^{-z} \cancel{dz} + e^{-z})\,dz + C$$

$$= i\left[\int -ze^{-z}\,dz + \int e^{-z}\,dz\right] + C$$

$$= i\left[-z\frac{e^{-z}}{(-1)} - (-1)\,e^{-z}/(-1)^2 + 0 + \frac{e^{-z}}{(-1)}\right] + C$$

$$= i\left[ze^{-z} + e^{-z} - e^{-z}\right] + C$$

$$\boxed{\int e^{ax}dx = \frac{e^{ax}}{a}}$$

$f(z) = ize^{-z} + C$. is required analytic function

⑦ Find the analytic function $f(z) = u + iv$ given $2u + v = e^x(\cos y - \sin y)$

Soln

Given $2u + v = e^x(\cos y - \sin y) \longrightarrow ①$

Diff ① partially w.r.t x

$2u_x + v_x = e^x(\cos y - \sin y)$

$2u_x(z,0) + v_x(z,0) = e^z(\cos 0 - \sin 0) = e^z$

~~Diff ① w.r.t~~ Using C-R eqn $v_x = -u_y$

~~$2u_x$~~ $2u_x(z,0) - u_y(z,0) = e^z \longrightarrow ②$

$$\boxed{u_x(z,0) = -\operatorname{cosec}^2 z}$$

$$u_y = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x\,[2\sinh 2y - 0]}{(\cosh 2y - \cos 2x)^2}$$

$$\boxed{u_y(z,0) = 0}$$

Formula ①

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + C$$

$$= \int -\operatorname{cosec}^2 z\,dz - i\int 0\,dz + C$$

$$= \int d(\cot z) + C$$

$$f(z) = \cot z + C.$$

⑥ Find the analytic function whose imaginary part is given by $v = e^{-x}(x\cos y + y\sin y)$.

Soln

Given $v = xe^{-x}\cos y + e^{-x}y\sin y$.

$$v_x = \cos y\left(x(-e^{-x}) + e^{-x}(1)\right) - e^{-x}y\sin y$$

$$v_x(z,0) = \cos 0\left(-ze^{-z} + e^{-z}\right) - e^{-z}(0)\sin(0)$$

$$\boxed{v_x(z,0) = -ze^{-z} + e^{-z}}$$

$$v_y = -xe^{-x}\sin y + e^{-x}\left[\sin y\,(1) + y\cos y\right]$$

$$v_y(z,0) = -ze^{-z}\sin(0) + e^{-z}\left[\sin(0) + 0.\cos 0\right]$$

$$\boxed{v_y(z,0) = 0}$$

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8 Find the analytic function for which $\frac{\sin 2x}{\cosh 2y - \cos 2x}$ is the real part. Hence determine the analytic function $u+iv$ for which $u+v$ is above function.

Soln :-

$$u+v = \frac{\sin 2x}{\cosh 2y - \cos 2x} \longrightarrow (A)$$

Diff (A) p.w.r. to 'x' we get

$$u_x + v_x = \frac{(\cosh 2y - \cos 2x)[2\cos 2x] - \sin 2x[2\sin 2x]}{(\cosh 2y - \cos 2x)^2}$$

$$u_x - u_y = \frac{(\cosh 2y - \cos 2x)[2\cos 2x] - \sin 2x[2\sin 2x]}{(\cosh 2y - \cos 2x)^2}$$

$$u_x(z,0) - u_y(z,0) = \frac{2\cos 2z(1 - \cos 2x) - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{-2}{1-\cos 2z}$$

$$= \frac{-2}{2\sin^2 z}$$

$$u_x(z,0) - u_y(z,0) = -\operatorname{cosec}^2 z. \longrightarrow (1)$$

Diff (A) p.w.r.t 'y' we get

$$u_y + v_y = \frac{0 - \sin 2x \; (\sinh 2y)}{(\cosh 2y - \cos 2x)^2} \; (2)$$

$$u_y + u_x = \frac{0 - \sin 2x \; (\sinh 2y)}{(\cosh 2y - \cos 2x)^2} \; (2)$$

$$u_y(z,0) + u_x(z,0) = 0. \longrightarrow (2)$$

$$(1) + (2) \Rightarrow 2u_x(z,0) = -\operatorname{cosec}^2 z.$$

$$u_x(z,0) = -\frac{1}{2}\operatorname{cosec}^2 z.$$

$$u_y(z,0) = \frac{1}{2}\operatorname{cosec}^2 z.$$

Let $w = f(z) = u + iv$.

$$f'(z) = u_x + iv_x.$$

$$= u_x - iu_y$$

$$f(z) = \int u_x(z,0)\,dy - i\int u_y(z,0)\,dz + C$$

$$= \int -\frac{1}{2}\operatorname{cosec}^2 z\,dz - i\int \frac{1}{2}\operatorname{cosec}^2 z + C$$

$$= \frac{1}{2}\cot z + i\,\frac{1}{2}\cot z + C$$

$$f(z). = \frac{1}{2}\cot z\,(1+i) + C.$$

Unit 4 COMPLEX INTEGRATION

Basic

① **Partial Fraction**

(i) $$\frac{f(z)}{(z-\alpha)(z-\beta)(z-\gamma)} = \frac{A}{(z-\alpha)} + \frac{B}{(z-\beta)} + \frac{C}{(z-\gamma)}$$

(ii) $$\frac{f(z)}{(z-a)^2(z-b)} = \frac{A}{(z-a)} + \frac{B}{(z-a)^2} + \frac{C}{(z-b)}$$

Here degree of numerator < degree of denominator.

② Suppose α and β are roots of $z^2+az+b=0$

then $z^2+az+b = (z-\alpha)(z-\beta) = 0$.

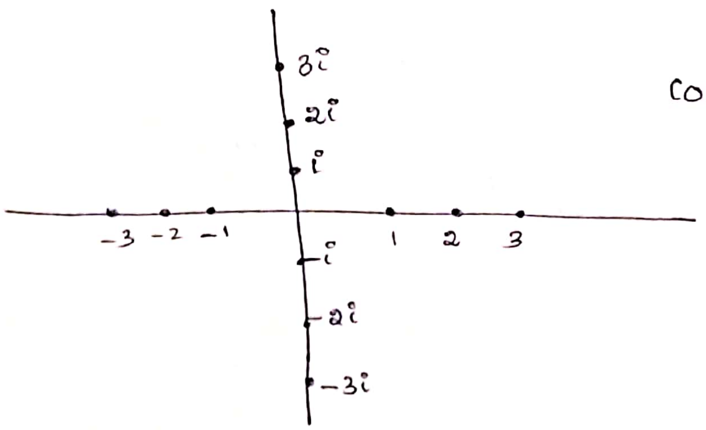
Example $z^2+3z+2 = 0$.

1	2
z	z

$\therefore (z^2+3z+2) = (z+1)(z+2) = 0$.

Here 1, 2 are roots of z^2+3z+2.

③



Complex plane Diagram

Cauchy Integral Formula

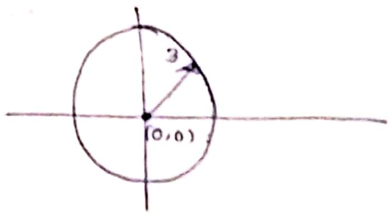
If $f(z)$ is analytic inside and on a simple closed curve C and if a is any point inside C then

$$\boxed{\int_C \frac{f(z)}{z-a}\,dz = 2\pi i \times f(a)}$$

In general
$$\boxed{\int_C \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(a)}$$
Here $f^{(n)}$ means n^{th} derivative of 'f'

Problems

① Evaluate $\displaystyle\int_C \frac{z^2}{(z-1)^2(z+2)}\,dz$ where C is $|z| = 3$



Soln $z=1$ (lies inside C)

$z=-2$ (lies inside C)

Partial Fraction : $\displaystyle\frac{z^2}{(z-1)^2(z+2)} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{(z+2)}$

$$= \frac{A(z-1)(z+2) + B(z+2) + C(z-1)^2}{(z-1)^2(z+2)}$$

$\Rightarrow z^2 = A(z-1)(z+2) + B(z+2) + C(z-1)^2 \longrightarrow ①$

Put $z=1$ in ①

$1 = B(1+2)$

$\boxed{1/3 = B}$

Put $z=-2$ in ①

$(-2)^2 = C(-2-1)^2$

$4 = 9C$

$\boxed{4/9 = C}$

Equating coeff of z^2 both sides

$1 = A + C$

$1 - C = A$

$1 - 4/9 = A$

$\boxed{5/9 = A}$

$$\therefore \frac{z^2}{(z-1)^2(z+2)} = \frac{5/9}{(z-1)} + \frac{1/3}{(z-1)^2} + \frac{4/9}{(z+2)}$$

$$\int_C \frac{z^2}{(z-1)^2(z+2)}\,dz = \int_C \frac{5/9}{(z-1)}\,dz + \int_C \frac{1/3}{(z-1)^2}\,dz + \int_C \frac{4/9}{(z+2)}\,dz$$

Cauchy integral Formula is

$$\boxed{\int_C \frac{f(z)\,dz}{(z-a)} = 2\pi i \times f(a)}$$

$$\boxed{\int_C \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^{n}(a)}$$

$$= 2\pi i\left(5/9\right) + \frac{2\pi i}{1!}\left(1^{st} \text{ derivative of } 1/3 \text{ at } 1\right) + 2\pi i\left(4/9\right)$$

$$= 2\pi i\left(5/9\right) + \frac{2\pi i}{1!}(0) + 2\pi i\left(4/9\right)$$

$$= 2\pi i\left(5/9 + 4/9\right)$$

$$= 2\pi i\left(9/9\right)$$

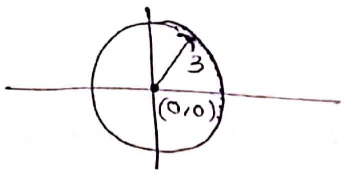
$$= 2\pi i.$$

(2) Evaluate $\displaystyle\int_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz$ where C is $|z| = 3$.

Soln.

$z = 1$ (lies inside C)

$z = 2$ (lies inside C)



Partial Fraction: $\displaystyle\frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$

$$\therefore \sin\pi z^2 + \cos\pi z^2 = A(z-2) + B(z-1) \longrightarrow (1)$$

Put $z = 1$ in (1)

$\sin\pi + \cos\pi = A(1-2)$

$0 - 1 = -A$

$\boxed{1 = A}$

Put $z = 2$ in (1)

$\sin 4\pi + \cos 4\pi = B(2-1)$

$0 + 1 = B$

$\boxed{1 = B}$

$$\therefore \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} = \frac{1}{(z-1)} + \frac{1}{(z-2)}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = \int_C \frac{1}{(z-1)}\, dz + \int_C \frac{1}{(z-2)}\, dz$$

Cauchy integral Formula is $\boxed{\int_C \frac{f(z)}{(z-a)}\, dz = 2\pi i \times f(a)}$

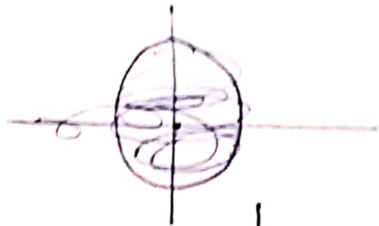
$$= (2\pi i \times 1) + (2\pi i \times 1)$$

$$= 4\pi i$$

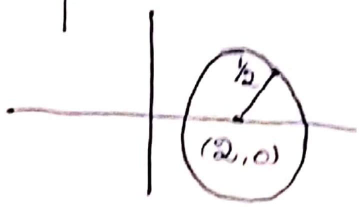
(3) Evaluate $\int_C \frac{z\, dz}{(z-1)(z-2)^2}$ where C is $|z-2| = \frac{1}{2}$ by cauchy integral formula

Soln

$z = 1$ (lies outside) $(\because |1-2| = 1 > \frac{1}{2})$



$z = 2$ (lies inside) $(\because |2-2| = 0 < \frac{1}{2})$



$$\therefore \int_C \frac{z}{(z-1)(z-2)^2}\, dz = \int_C \frac{\left(\frac{z}{z-1}\right)}{(z-2)^2}\, dz$$

(Here we take $(z-1)$ term in numerator)

Cauchy integral formula $\boxed{\int_C \frac{f(z)\, dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^n(a)}$

$$= \frac{2\pi i}{1!} \left[\text{First derivative of } \left(\frac{z}{z-1}\right) \text{ at } z = 2\right]$$

$$= 2\pi i \left[\frac{(z-1)(1) - z(1-0)}{(z-1)^2} \right]_{\text{Put } z=2}$$

$$= 2\pi i \left[\frac{-1}{(z-1)^2} \right]_{\text{Put } z=2}$$

$$= 2\pi i \left[\frac{-1}{(2-1)^2} \right]$$

$$= 2\pi i(-1)$$

$$= -2\pi i.$$

(4) Evaluate $\displaystyle\int_C \frac{z+4}{z^2+2z+5}\,dz$ where C is $|z+1+i| = 2$.

using cauchy integral formula

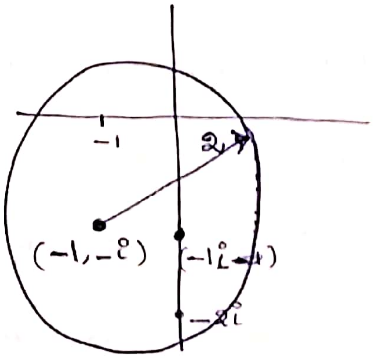
Soln

Given $|z+1+i| = 2$

$|z-(-1-i)| = 2$

Centre $= (-1-i)$

radius $= 2$



$$z^2+2z+5 = 0. \Rightarrow z = \frac{-2 \pm \sqrt{4-4(1)(5)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{-16}}{2}$$

$$= \frac{-2 \pm i4}{2}$$

$\sqrt{-1} = i$

$$= -1 \pm 2i$$

$\alpha = -1+2i$ (lies outside) $(\because |-1+2i+1+i| = |3i| = \sqrt{9} > 2$

$\beta = -1-2i$ (lies inside) $(\because |-1-2i+1+i| = |-i| = \sqrt{1} < 2$

$\therefore\ z^2+2z+5 = (z-\alpha)(z-\beta)$

$$\therefore \int_C \frac{(z+4)}{(z^2+2z+5)}\,dz = \int_C \frac{(z+4)}{(z-\alpha)(z-\beta)}\,dz$$

$$= \int_C \frac{\left(\frac{z+4}{(z-\alpha)}\right)}{(z-\beta)}\,dz$$

$$= 2\pi i \left[\frac{z+4}{z-\alpha}\right]$$

Put $z=\beta$

$$= 2\pi i \left(\frac{\beta+4}{\beta-\alpha}\right)$$

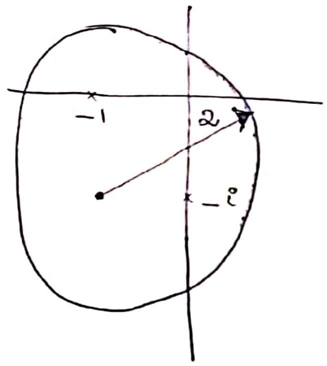
$$= \frac{(2\pi i)(-1-2i+4)}{(-1-2i)-(-1+2i)}$$

$$= \frac{(2\pi i)(-1-2i+4)}{-4i}$$

$$= -\pi/2\,(3-2i)$$

(5) Evaluate $\int_C \frac{(z+1)}{(z^2+2z+4)}\,dz$ where C is the circle

$|z+1+i| = 2$.



Soln: Given C is $|z+1+i| = 2$

$|z-(-1-i)| = 2$

centre $= (-1-i)$

radius $= 2$

$z^2+2z+4 = 0 \quad \Rightarrow \quad z = \frac{-2 \pm \sqrt{4-4(1)(4)}}{2(1)}$

$$z = \frac{-2 \pm \sqrt{-12}}{2}$$

$$x = \frac{-2 \pm 2\sqrt{3}\,i}{2}$$

$$z = -1 \pm \sqrt{3}\,i.$$

$\alpha = -1+\sqrt{3}i$ (lies outside) $\left(\because \left|-1+\sqrt{3}i+1+i\right| > 2\right.$
$= \sqrt{(1+\sqrt{3})^2} > 2$

$\beta = (-1-\sqrt{3}i)$ (lies inside) $\left(\because \left|-1-\sqrt{3}i+1+i\right| = \sqrt{(1-\sqrt{3})^2}\right|$
$\left.= \sqrt{1+3-2\sqrt{3}} < 2\right)$

$\therefore \quad z^2+2z+4 = (z-\alpha)(z-\beta)$

$$\therefore \int_C \frac{z+1}{z^2+2z+4}\,dz = \int_C \frac{(z+1)}{(z-\alpha)(z-\beta)}\,dz$$

$$= \int_C \frac{\left(\frac{z+1}{z-\alpha}\right)}{(z-\beta)}\,dz$$

$$= 2\pi i \times \left[\frac{z+1}{z-\alpha}\right]_{\text{put } z=\beta}$$

$$= 2\pi i \times \left[\frac{\beta+1}{\beta-\alpha}\right]$$

$$= 2\pi i \times \left[\frac{-1-\sqrt{3}i+1}{(-1-\sqrt{3}i)-(-1+\sqrt{3}i)}\right]$$

$$= 2\pi i \times \left(\frac{-\sqrt{3}i}{-2\sqrt{3}i}\right)$$

$$= \pi i.$$

Problems (Based on Cauchy's Integral formula)

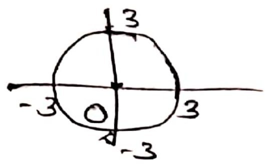
① Evaluate $\int_C \frac{z}{z-2}\,dz$ where C is $|z|=3$.

Soln:-

Cauchy's integral formula

$$\boxed{\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)} \longrightarrow ①$$

Given $f(z) = z$, $a = 2$, C is $|z|=3$



$a=2$ lies inside $|z|=3$.

$\therefore f(a) = f(2) = 2$.

$\therefore ① \Rightarrow \int_C \frac{z}{z-2}\,dz = 2\pi i \times 2 = 4\pi i$.

② Evaluate $\int_C \frac{z}{(z-1)^3}\,dz$ where C is $|z|=2$ using Cauchy's integral formula.

Soln

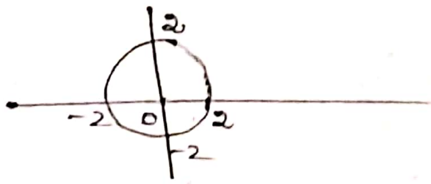
We know that Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{n}(a). \longrightarrow ①$$

Given $f(z) = z$

$a = 1$

$c = (|z| = 2)$



$a = 1$ lies inside $|z| = 2$.

$\therefore$ From ①

$$\cancel{\int_C \frac{f(z)}{(z-a)^3}}$$

$$\int_C \frac{z}{(z-1)^3}\, dz = \frac{2\pi i}{2!} f^{(2)}(1)$$

$$= \frac{2\pi i}{2} f''(1)$$

$$= \pi i \times 0$$

$$= 0.$$

③ Evaluate $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}$ using Cauchy's integral formula where C is $|z| = \frac{3}{2}$.

Soln

Cauchy's integral formula

$$\boxed{\int_C \frac{f(z)}{z-a} = 2\pi i \cdot f(a)} \longrightarrow ①$$

Given $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\, dz$ where C is $|z| = 3/2$.

$$f(z) = \frac{\cos \pi z^2}{(z-2)}, \quad a = 1$$

$z = 1$ lies inside $|z| = 3/2$.

$$f(1) = \frac{\cos \pi}{1-2} = 1.$$

$$\therefore \int_C \frac{\cos \pi z^2}{(z-2)(z-1)} dz = 2\pi i \times f(1)$$

$$= 2\pi i$$

(4) Evaluate $\int_C \frac{z^2}{(z-1)^2 (z+2)} dz$ where C is $|z| = 3$.

Soln:-

Consider $$\frac{z^2}{(z-1)^2(z+2)} = \frac{A}{(z-1)} + \frac{B}{(z-1)^2} + \frac{C}{z+2}$$

$$= \frac{A(z-1)(z+2) + B(z+2) + C(z-1)^2}{(z-1)^2(z+2)}$$

$$\therefore z^2 = A(z-1)(z+2) + B(z+2) + C(z-1)^2 \longrightarrow (1)$$

Put $z = 1$

$1 = 3B$

$\boxed{B = 1/3}$

Put $z = -2$

$4 = C(9)$

$\boxed{C = 4/9}$

Equating coeffs of z^2

$1 = A + C$

$A = 1 - C$

$A = 1 - 4/9$

$\boxed{A = 5/9}$

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$$\therefore \quad f(z) = \frac{z^2}{(z-1)^2(z+2)} = \frac{5}{9}\left(\frac{1}{z-1}\right) + \frac{1}{3}\left(\frac{1}{z-1}\right)^2 + \frac{4}{9}\left(\frac{1}{z+2}\right).$$

$$\int_C f(z)\,dz = \int_C \frac{5/9}{(z-1)}\,dz + \int_C \frac{1/3}{(z-1)^2} + \int_C \frac{4/9}{(z+2)}.$$

Clearly $z=1$ lies inside $|z|=3$

$z=-2$ lies inside $|z|=3$.

$$= \frac{5}{9}\,2\pi i\, f(1) + \frac{1}{3}\,2\pi i\, f'(1) + \frac{4}{9}\,2\pi i\, f(-2)$$

$$= \left(\frac{5}{9} \times 2\pi i \times 1\right) + \left(\frac{1}{3} \times 2\pi i \times 0\right) + \left(\frac{4}{9} \times 2\pi i \times +1\right)$$

$$= \frac{5}{9}\,2\pi i + \frac{4}{9}\,2\pi i$$

$$= 2\pi i\,(1)$$

$$= 2\pi i.$$

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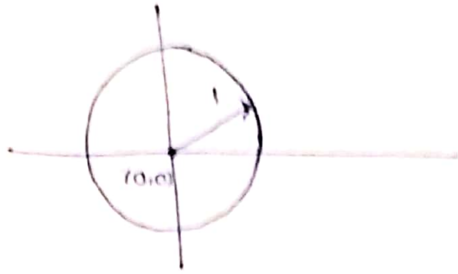
Contour Integration

Type (1) $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_C f(z)\, dz$

Diagram



C is circle with
center = (0,0)
radius = 1

Put $z = e^{i\theta}$

$\Rightarrow dz = i e^{i\theta} \cdot d\theta$

$= iz\, d\theta$

$\Rightarrow d\theta = \dfrac{dz}{iz}$

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$= \dfrac{z + 1/z}{2}$

$\cos\theta = \dfrac{z^2+1}{2z}$

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

$\sin\theta = \dfrac{z - 1/z}{2i}$

$\sin\theta = \dfrac{z^2-1}{2iz}$

$\therefore$

$$\boxed{\begin{aligned} z &= e^{i\theta} \\ d\theta &= \frac{dz}{iz} \\ \sin\theta &= \frac{z^2-1}{2iz} \\ \cos\theta &= \frac{z^2+1}{2z} \end{aligned}}$$

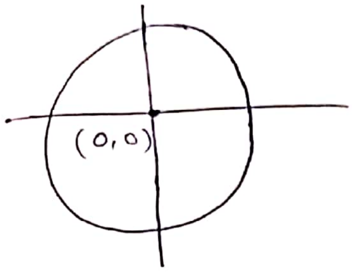
Remember

Problem (1) Evaluate $\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta$ by contour integration

Soln :-

$$\boxed{\begin{aligned} z &= e^{i\theta} \\ d\theta &= dz/iz \\ \cos\theta &= \frac{z^2+1}{2z} \\ \sin\theta &= \frac{z^2-1}{2iz} \end{aligned}}$$

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta = \int_C \frac{1}{2+\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$ where C is the circle with centre (0,0) and radius 1



$$= \int_C \frac{1}{\left(\frac{4z+z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$= \int_C \frac{2z}{(4z+z^2+1)} \frac{dz}{iz}$$

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta = \frac{1}{i} \int_C \frac{2}{z^2+4z+1}\, dz \longrightarrow (1)$$

Apply cauchy residue thm for $\int_C \frac{2}{z^2+4z+1} dz$.

$$f(z) = \frac{2}{z^2+4z+1}.$$

To find poles

$$z^2+4z+1 = 0.$$

$$z = \frac{-4 \pm \sqrt{16-4(1)(1)}}{2(1)}$$

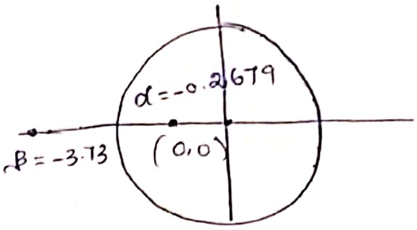
$$= \frac{-4 \pm \sqrt{12}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}}{2}$$

$z = -2 \pm \sqrt{3}$

Take $\alpha = -2+\sqrt{3} = -0.2679$ (lies inside c)

$\beta = -2-\sqrt{3} = -3.7320$ (lies outside c)



$\therefore z^2+4z+1 = (z-\alpha)(z-\beta)$.

$$\text{Res}\left[f(z);\ z=\alpha\right] = \lim_{z\to\alpha}\left[(z-\alpha)\, f(z)\right]$$

$$= \lim_{z\to\alpha}\left[(z-\alpha)\frac{2}{(z^2+4z+1)}\right]$$

$$= \lim_{z\to\alpha}\left[\cancel{(z-\alpha)}\frac{2}{\cancel{(z-\alpha)}(z-\beta)}\right]$$

$$= \lim_{z\to\alpha}\left[\frac{2}{z-\beta}\right]$$

$$= \frac{2}{\alpha-\beta}.$$

$$= \frac{2}{(-2+\sqrt{3})-(-2-\sqrt{3})}$$

$$= \frac{2}{-2+\sqrt{3}+2+\sqrt{3}}$$

$$= \frac{2}{2\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}.$$

$\therefore$ By cauchy residue thm $\displaystyle\int_C \frac{2\,dz}{z^2+4z+1} = 2\pi i\,(\text{Sum of residues})$

$= 2\pi i\left(\frac{1}{\sqrt{3}}\right) \longrightarrow (2)$

Sub (2) in (1)

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\,d\theta = \frac{1}{i} \times 2\pi i\left(\frac{1}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}.$$

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(2) Evaluate $\int_0^{2\pi} \frac{1}{13+5\cos\theta}\, d\theta$ using contour integration

Soln:

$$\boxed{\begin{aligned} z &= e^{i\theta} \\ d\theta &= dz/iz \\ \cos\theta &= \frac{z^2+1}{2z} \end{aligned}}$$

$$\int_0^{2\pi} \frac{1}{13+5\cos\theta}\, d\theta = \int_C \frac{1}{13+5\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

where C is circle with center $(0,0)$, radius $=1$

$$= \int_C \frac{2z}{13(2z)+5z^2+5} \frac{dz}{iz}$$

$$= \frac{1}{i}\int_C \frac{2}{5z^2+26z+5}$$

$$= \frac{1}{5i} \int_C \frac{2}{z^2+\left(\frac{26}{5}\right)z+1} \longrightarrow (1)$$

Apply cauchy residue theorem for $\int_C \frac{2}{z^2+\left(\frac{26}{5}\right)z+1}$.

$$f(z) = \frac{2}{z^2+\left(\frac{26}{5}\right)z+1}.$$

To find poles

$$z^2+\left(26/5\right)z+1 = 0$$

$$a=1, \quad b = 26/5, \quad c=1.$$

$$z = \frac{-b \pm\sqrt{b^2-4ac}}{2a}$$

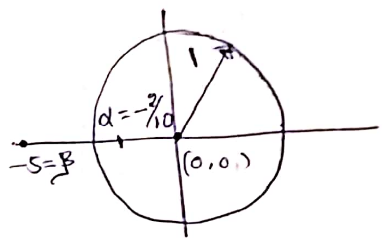
$$= \frac{-26/5 \pm \sqrt{\left(26/5\right)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-26/5 \pm \sqrt{\frac{676}{25} - 4}}{2}$$

$$= \frac{-26/5 \pm \sqrt{\frac{676-100}{25}}}{2}$$

$$= \frac{-26/5 \pm \sqrt{\frac{576}{25}}}{2}$$

$$= \frac{\left(-\frac{26}{5} \pm \frac{24}{5}\right)}{2}$$



$$z = \frac{-26 \pm 24}{10}$$

$$\alpha = \frac{-26+24}{10} = -2/10 \quad \text{(lies inside C)}$$

$$\beta = \frac{-26-24}{10} = -5 \quad \text{(lies outside C)}$$

$$f(z) = 2/(z-\alpha)(z-\beta)$$

$$\text{Res}\left[f(z);\, z=\alpha\right] = \lim_{z\to\alpha}\left[(z-\alpha)\, f(z)\right]$$

$$= \lim_{z\to\alpha}\left[(z-\alpha)\frac{2}{(z-\alpha)(z-\beta)}\right]$$

$$= \lim_{z\to\alpha}\left[\frac{2}{z-\beta}\right]$$

$$= \frac{2}{\alpha-\beta}$$

$$= \frac{2}{-2/10+5}$$

By cauchy residue thm,

$$= 10\times 2/48 = 5/12$$

$$\therefore \int_C \frac{2}{z^2+\left(\frac{26}{5}\right)z+1}\, dz = 2\pi i\,(\text{sum of residues})$$

$$= 2\pi i\left(5/12\right) \longrightarrow (2)$$

sub (2) in (1)

$$\int_0^{2\pi} \frac{1}{13+5\cos\theta}\, d\theta = \frac{1}{5i} \times 2\pi i \times 5/12 = \frac{\pi}{6}$$

(3) Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \frac{\pi}{8}$ using contour integration

Soln

$e^{i3\theta} = \cos 3\theta + i\sin 3\theta$

$\therefore \cos 3\theta =$ Real part of $e^{i3\theta}$

$$\boxed{\begin{array}{l} z = e^{i\theta} \\ d\theta = dz/iz \\ \cos\theta = \frac{z^2+1}{2z} \end{array}} \Rightarrow z^3 = (e^{i\theta})^3 = e^{i3\theta}$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} d\theta = \int_0^{2\pi} \frac{R.P \text{ of } e^{i3\theta}}{5-4\cos\theta} d\theta$$

$$= R.P \int_0^{2\pi} \frac{e^{i3\theta}}{5-4\cos\theta} d\theta$$

$$= R.P \int_C \frac{z^3}{5-4\left(\frac{z^2+1}{2z}\right)} \frac{dz}{iz}$$

$$= R.P \left[\frac{1}{i} \int_C \frac{z^3 \times 2z}{10z-4z^2-4} \frac{dz}{z} \right]$$

$$= R.P \left[\frac{2}{-i} \int_C \frac{z^3}{4z^2-10z+4} dz \right]$$

$$= R.P \left[-\frac{2}{i\times 4} \int_C \frac{z^3}{(z^2-5/2 z+1)} dz \right]$$

$$= R.P \left[\frac{-1}{2i} \int_C \frac{z^3}{z^2-5/2 z+1} dz \right] \quad \text{--- (1)}$$

Appy cauchy residue theorem for

$$\int_C \frac{z^3}{z^2-5/2 z+1}$$

$$f(z) = \frac{z^3}{z^2 - \frac{5}{2}z + 1}$$

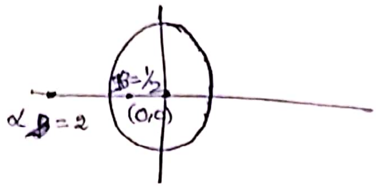
To find poles

$z^2 - \frac{5}{2}z + 1 = 0$, $\quad (a = 1,\ b = -\frac{5}{2},\ c = 1)$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4}}{2} = \frac{\frac{5}{2} \pm \sqrt{\frac{9}{4}}}{2}$$

$$z = \frac{\frac{5}{2} \pm \frac{3}{2}}{2} = \frac{5 \pm 3}{4}$$

$\alpha = \frac{5+3}{4} = 2$ (lies outside C)



$\beta = \frac{5-3}{4} = \frac{1}{2}$ (lies inside C)

$\therefore\ z^2 - \frac{5}{2}z + 1 = (z - \alpha)(z - \beta)$

$$\text{Res}\left[f(z)\,;\, z = \beta\right] = \lim_{z \to \beta}\left[(z-\beta)\cdot f(z)\right] = \lim_{z\to\beta}\left[(z-\beta)\frac{z^3}{(z-\alpha)(z-\beta)}\right]$$

$$= \lim_{z\to\beta}\left[\frac{z^3}{z-\alpha}\right]$$

$$= \frac{\beta^3}{\beta - \alpha} = \frac{(\frac{1}{8})}{(\frac{1}{2} - 2)}$$

$$= \frac{(\frac{1}{8})}{-\frac{3}{2}}$$

$$= \frac{2}{-24}$$

$$\text{Res}\left[f(z)\,;\, z = \beta\right] = -\frac{1}{12}$$

$\therefore$ By cauchy residue theorem, $\displaystyle\int_C \frac{z^3}{z^2 - \frac{5}{2}z + 1}\,dz = 2\pi i\left(-\frac{1}{12}\right) \longrightarrow (2)$

Sub (2) in (1)

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\,d\theta = R.P\left[-\frac{1}{2i} \times 2\pi i \times \left(-\frac{1}{12}\right)\right] = R.P\left[\frac{\pi}{12}\right]$$

$$= \frac{\pi}{12}$$

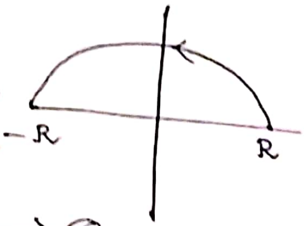
Contour Integration

Problem Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$ using contour integration. Hence find $\int_{0}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$

Soln.

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \int_C \frac{z^2}{(z^2+a^2)(z^2+b^2)}\,dz$$

where C is upper half of semicircle with diameter from $(-R, R)$



$$= 2\pi i \times \text{Sum of residues} \longrightarrow (1).$$

$$f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}.$$

To find poles

$$(z^2+a^2)(z^2+b^2) = 0.$$

$$(z+ia)(z-ia)(z+ib)(z-ib) = 0.$$

$$\left.\begin{array}{l} z = -ia \\ z = -ib \end{array}\right\} \text{lies outside } C$$

$$\left.\begin{array}{l} z = ia \\ z = ib. \end{array}\right\} \text{lies inside } C.$$

To find Residues

$$Res\left[f(z); z = a\right] = \lim_{z \to a} (z-a).\, f(z).$$

At $z = ia$

$$Res\left[f(z); z = ia\right] = \lim_{z \to ia} \left[\cancel{(z-ia)} \frac{z^2}{(z+ia)\cancel{(z-ia)}(z+ib)(z-ib)} \right]$$

$$= \frac{(ia)^2}{(2ia)(ia+ib)(ia-ib)}$$

$$= \frac{-a^2}{2ia(i)^2\left[a^2-b^2\right]}$$

$$= \frac{a^2}{2ia\left[a^2-b^2\right]}.$$

At $z = ib$

$$\text{Res}\left[f(z) ; z = ib\right] = \lim_{z \to ib}\left[(z-ib)\frac{z^2}{(z+ia)(z-ia)(z+ib)(z-ib)}\right]$$

$$= \frac{(ib)^2}{(ib+ia)(ib-ia)\,2ib}$$

$$= \frac{-b^2}{i^2(b+a)(b-a)\,2ib}$$

$$= \frac{b^2}{-(a^2-b^2)\,2ib}$$

$$= \frac{-b^2}{(a^2-b^2)\,2ib}.$$

$\therefore$ From ①

$$\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = 2\pi i\left[\frac{a^2}{2ia(a^2-b^2)} - \frac{b^2}{2ib(a^2-b^2)}\right]$$

$$= \cancel{\frac{2\pi i}{2i}\left[\frac{(a^2-b^2)}{}\right]}$$

$$= \frac{2\pi i}{2i(a^2-b^2)}\left[a-b\right]$$

$$= \frac{\pi}{(a+b)}.$$

$$\int_{0}^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{1}{2}\times\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$$

$$= \frac{\pi}{2(a+b)}.$$

$$\text{Res}\left[f(z),\ z=bi\right] = \lim_{z\to bi}\left[\frac{(z-bi)\cdot e^{iz}}{(z-ai)(z+ai)(z-bi)(z+bi)}\right]$$

$$= \frac{e^{i(bi)}}{(bi-ai)(bi+ai)(2bi)}$$

$$= \frac{e^{-b}}{i(b-a)\,i(b+a)\,2bi}$$

$$= \frac{e^{-b}}{(a^2-b^2).\,2bi}.$$

$$\therefore \int_C f(z).\,dz = 2\pi i\left[\frac{e^{-b}}{(a^2-b^2)2bi} - \frac{e^{-a}}{(a^2-b^2)2ai}\right]$$

$$= \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right].$$

$$\therefore \int_{-\infty}^{\infty} f(x).\,dx = \int_C f(z).\,dz = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$\int_{-\infty}^{\infty} f(x)dx = 2\int_0^{\infty} f(x).\,dx$$

$$\therefore \int_0^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

Type (3) :- Integrals of the form $\int_{-\infty}^{\infty} f(x).\cos mx\,dx$

(or) $\int_{-\infty}^{\infty} f(x).\sin mx\,dx$ where $f(x) \to 0$ as $x \to \infty$

Problems

1 Evaluate $\int_{0}^{\infty} \frac{\cos ax}{x^2+1}\,dx$, $a > 0$.

Soln

Let $f(x) = \frac{e^{iax}}{x^2+1}$.

$$f(z) = \frac{e^{iaz}}{z^2+1} = \frac{e^{iaz}}{(z+i)(z-i)}.$$

$\therefore \int_{-\infty}^{\infty} f(x).dx = \int_{C} f(z).dz$ where C is semicircle $|z| = R$ with bounding diameter $[-R, R]$.

To find Poles of $f(z)$

$$z^4+10z^2+9=0.$$

Put $z^2=t$

$$\therefore\ t^2+10t+9=0$$

$$(t+1)(t+9)=0$$

$$t=-1,\quad t=-9$$

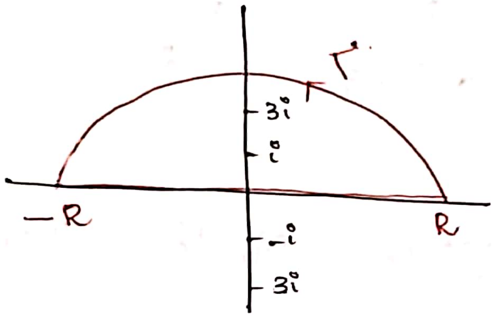
$$z^2=-1,\quad z^2=-9$$

$$z=\pm i,\quad z=\pm 3i.$$

9

$$\begin{array}{c|c} 1 & 9 \\ \hline t & t \end{array}$$

$\therefore$ Poles are $i, -i, 3i, -3i$



$i, 3i \longrightarrow$ lies inside

$-i, -3i \longrightarrow$ lies outside.

$$\therefore \int_{-\infty}^{\infty} f(z)\,dz = 2\pi i\left[\text{sum of residues}\right].$$

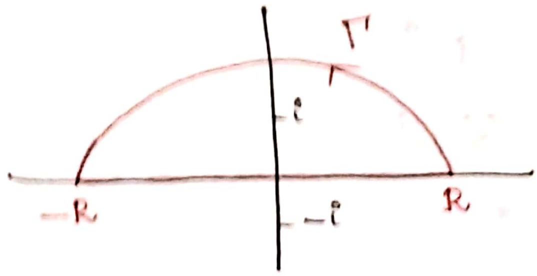
$$\text{Res}\left[f(z), z=i\right] = \lim_{z\to i}\left[(z-i)\times\frac{z^2-z+2}{(z-i)(z+i)(z-3i)(z+3i)}\right]$$

$$=\frac{-1-i+2}{2i(-2i)(4i)} = \frac{-i+1}{16i}$$

$$\text{Res}\left[f(z), z=3i\right] = \lim_{z\to 3i}\left[\frac{(z-3i).(z^2-z+2)}{(z-i)(z+i)(z-3i)(z+3i)}\right]$$

$$=\frac{-9-3i+2}{(2i)(4i)(6i)} = \frac{-3i-7}{-48i} = \frac{3i+7}{48i}.$$

$$\int_C f(z)\cdot dz = 2\pi i \left[\text{Sum of residues}\right]$$



Poles are $i, -i$

i lies inside

$-i$ lies outside

$$\therefore \quad \operatorname{Res}\left[f(z), z=i\right] = \lim_{z \to i}\left[(z-i)\cdot\frac{e^{iaz}}{(z+i)(z-i)}\right]$$

$$= \frac{e^{ia \times i}}{i+i}$$

$$= \frac{e^{-a}}{2i}.$$

$$\therefore \quad \int_C f(z)\cdot dz = 2\pi i\left[\frac{e^{-a}}{2i}\right] = \pi e^{-a}.$$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)\cdot dx = \int_C f(z)\cdot dz = \pi e^{-a}$$

$$\text{(i)} \quad \int_{-\infty}^{\infty} f(x)\cdot dx = \pi e^{-a}$$

$$\int_{-\infty}^{\infty} f(x)\cdot dx = 2\int_0^{\infty} f(x)\cdot dx \qquad (\because f(x) \text{ is even})$$

$$\therefore \quad 2\int_0^{\infty} f(x)\cdot dx = \pi e^{-a}$$

$$\therefore \quad \int_0^{\infty} f(x)\cdot dx = \frac{\pi e^{-a}}{2}.$$

$$\int_0^{\infty} \frac{\cos ax}{x^2+1}\, dx = \frac{\pi e^{-a}}{2}.$$

(2) Show that $\int_0^\infty \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{2(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$.

Soln :-

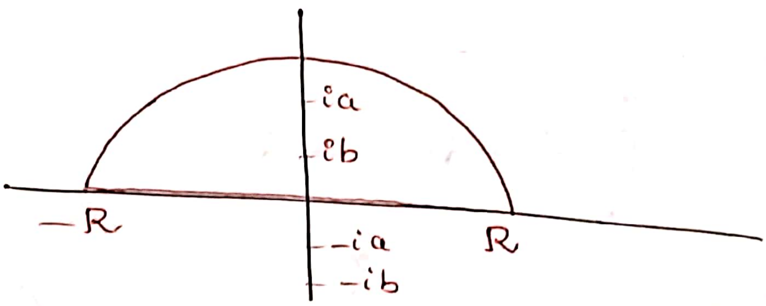
$$f(x) = \frac{\cancel{\cos x}\; e^{iax}}{(x^2+a^2)(x^2+b^2)}$$

$$f(z) = \frac{e^{iaz}}{(z^2+a^2)(z^2+b^2)}$$

$$f(z) = \frac{e^{iaz}}{(z+ia)(z-ia)(z+ib)(z-ib)}$$

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \int_C f(z)\,dz$ where C is the semicircle $|z| = R$ with bounding diameter $[-R, R]$.

$$\int_C f(z)\,dz = 2\pi i\,[\text{sum of residues}].$$



Poles are ia, ib
$-ia, -ib$

$ia, ib \rightarrow$ lies inside

$-ia, -ib \rightarrow$ lies outside.

$$\therefore \text{Res}[f(z), z = ia] = \lim_{z\to ia}\left[\frac{(z-ia)\cdot e^{iaz}}{(z+ia)(z-ia)(z+ib)(z-ib)}\right]$$

$$= \frac{e^{ia(ia)}}{(2ai)\,i(a+b)\,i(a-b)} = \frac{e^{-a}}{-2ai\,(a^2-b^2)}$$

Laurent's Series

Formula 1) $(1-x)^{-1} = \sum_{n=0}^{\infty} (x)^n$

2) $(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$

Problem 1 Expand $f(z) = \dfrac{1}{z^2+4z+3}$ as a Laurent's Series valid in the regions (i) $1 < |z| < 3$. and (ii) $|z| > 3$.

Soln

$$f(z) = \frac{1}{z^2+4z+3} = \frac{1}{(z+1)(z+3)}$$

Partial Fraction

$$\frac{1}{(z+1)(z+3)} = \frac{A}{(z+1)} + \frac{B}{(z+3)} = \frac{A(z+3)+B(z+1)}{(z+1)(z+3)}$$

$$\Rightarrow \quad 1 = A(z+3) + B(z+1) \longrightarrow (1)$$

Put $z=-1$ in (1)

$1 = A(-1+3)$

$1 = 2A$

$\boxed{A = 1/2}$

Put $z=-3$ in (1)

$1 = B(-3+1)$

$1 = B(-2)$

$\boxed{B = -1/2}$

$$\therefore f(z) = \frac{A}{(z+1)} + \frac{B}{(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)} \longrightarrow (2)$$

(i) Given region : $1 < |z| < 3$

$\Rightarrow \quad 1 < |z|$ and $|z| < 3$

$\Rightarrow \quad \left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$.

From (2)

$$f(z) = \frac{1}{2z(1+1/z)} - \frac{1}{6(1+z/3)} = \frac{1}{2z}(1+1/z)^{-1} - \frac{1}{6}(1+z/3)^{-1}$$

$$= \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

(ii) $f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$

Given region $|z| > 3$.

$\Rightarrow \quad 3 < |z|$.

$\Rightarrow \quad 1 < 3 < |z|$.

$\therefore \quad 1 < |z|$ and $3 < |z|$

$\Rightarrow \quad \left|\frac{1}{z}\right| < 1$ and $\left|\frac{3}{z}\right| < 1$.

$$\therefore f(z) = \frac{1}{2z(1+1/z)} - \frac{1}{2z(1+3/z)}$$

$$= \frac{1}{2z}(1+1/z)^{-1} - \frac{1}{2z}(1+3/z)^{-1}$$

$$= \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n(1/z)^n - \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n(3/z)^n.$$

Problem Expand $f(z) = \dfrac{7z-2}{z(z+1)(z-2)}$ as a Laurent's series

valid in the region $1 < |z+1| < 3$

Soln

Given $f(z) = \dfrac{7z-2}{z(z+1)(z-2)}$

Partial Fraction

$$f(z) = \frac{7z-2}{z(z+1)(z-2)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-2}$$

$$= \frac{A(z+1)(z-2) + Bz(z-2) + Cz(z+1)}{z(z+1)(z-2)}$$

$$\Rightarrow 7z-2 = A(z+1)(z-2) + Bz(z-2) + Cz(z+1) \longrightarrow (1)$$

Put $z=0$

$-2 = A(1)(-2)$

$A = -2/-2$

$\boxed{A=1}$

Put $z=-1$

$-7-2 = B(-1)(-3)$

$-9 = 3B$

$B = -9/3$

$\boxed{B=-3}$

Put $z=2$

$14-2 = C(2)(3)$

$12 = 6C$

$C = 12/6$

$\boxed{C=2}$

$$\therefore f(z) = \frac{1}{z} - \frac{3}{z+1} + \frac{2}{z-2}$$

Given region : $1 < |z+1| < 3$.

Take $z+1 = u$

$\Rightarrow z = u-1$.

$\Rightarrow z-2 = u-1-2 = u-3$.

$$\therefore f(z) = \frac{1}{u-1} - \frac{3}{u} + \frac{2}{u-3}$$

Now $\quad 1 < |u| < 3.$

$\Rightarrow \quad 1 < |u| \quad$ and $\quad |u| < 3.$

$\Rightarrow \quad \left|\frac{1}{u}\right| < 1 \quad$ and $\quad \left|\frac{u}{3}\right| < 1.$

$$f(z) = \frac{1}{u-1} - \frac{3}{u} + \frac{2}{u-3}$$

$$= \frac{1}{u(1-1/u)} - \frac{3}{u} + \frac{2}{3(-1+u/3)}$$

$$= \frac{1}{u(1-1/u)} - 3/u - \frac{2}{3(1-u/3)}$$

$$\boxed{(1-x)^{-1} = \sum_{n=0}^{\infty} x^n.}$$

$$= \frac{1}{u}\left[1-1/u\right]^{-1} - 3/u - \frac{2}{3}\left[1-u/3\right]^{-1}$$

$$= \frac{1}{u}\sum_{n=0}^{\infty}\left(\frac{1}{u}\right)^n - 3/u - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{u}{3}\right)^n$$

$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{3}{(z+1)} - 2/3\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n.$$

Problem Expand $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ as a Laurents Series valid in the region $2 < |z| < 3$.

Soln

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = \frac{z^2-1}{z^2+5z+6}$$

$$\begin{array}{r|l} & 1 \\ \hline z^2+5z+6 & z^2-1 \\ & \underset{(-)}{z^2} \underset{(-)}{+5z} \underset{(-)}{+6} \\ \hline & -5z-7 \end{array}$$

$$\therefore \quad z^2-1 = (z^2+5z+6) - 5z - 7$$

$$\Rightarrow \quad z^2-1 = (z^2+5z+6) - (5z+7)$$

$$\Rightarrow \quad \frac{z^2-1}{z^2+5z+6} = 1 - \frac{(5z+7)}{z^2+5z+6}$$

$$\therefore \quad f(z) = 1 - \frac{(5z+7)}{z^2+5z+6} \longrightarrow (1)$$

Partial Fraction:

$$\frac{-(5z+7)}{(z^2+5z+6)} = \frac{-(5z+7)}{(z+2)(z+3)} = \frac{A}{(z+2)} + \frac{B}{(z+3)}$$

$$= \frac{A(z+3) + B(z+2)}{(z+2)(z+3)}$$

$$\Rightarrow \quad -(5z+7) = A(z+3) + B(z+2) \longrightarrow (2)$$

Put $z = -2$

$$-[5(-2)+7] = A(-2+3)$$

$$-[-10+7] = A$$

$$-[-3] = A$$

$$\boxed{A = 3}$$

Put $z = -3$

$$-[-15+7] = B(-3+2)$$

$$-[-8] = -B.$$

$$\boxed{B = -8}$$

$$\therefore \frac{-(5z+7)}{(z+2)(z+3)} = \frac{3}{(z+2)} - \frac{8}{(z+3)} \rightarrow ③$$

Sub ③ in ②

$$f(z) = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)}$$

Given region : $2 < |z| < 3$.

$$\Rightarrow \quad 2 < |z| \quad \text{and} \quad |z| < 3$$

$$\Rightarrow \quad \left|\frac{2}{z}\right| < 1 \quad \text{and} \quad \left|\frac{z}{3}\right| < 1.$$

$$\therefore f(z) = 1 + \frac{3}{z(1+2/z)} - \frac{8}{3(1+z/3)}$$

$$= 1 + \frac{3}{z}(1+2/z)^{-1} - \frac{8}{3}(1+z/3)^{-1}$$

$$\boxed{(1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n.}$$

$$= 1 + \frac{3}{z} \sum_{n=0}^{\infty} (-1)^n (2/z)^n - \frac{8}{3} \sum_{n=0}^{\infty} (-1)^n (z/3)^n.$$

5

Expand as a Laurent's Series for the function

$f(z) = \dfrac{z}{z^2-3z+2}$ in the following regions.

(i) $|z|<1$ (ii) $1<|z|<2$ (iii) $|z|>2$ (iv) $|z-1|<1$.

Soln :-

$$f(z) = \frac{z}{z^2-3z+2}$$

Partial fraction :-

$$f(z) = \frac{z}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$= \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$\Rightarrow \quad z = A(z-2) + B(z-1) \longrightarrow (1)$$

Put $z=1$ in (1)

$1 = A(1-2)$

$1 = -A$

$\boxed{A=-1}$

Put $z=2$ in (1)

$2 = B(2-1)$

$\boxed{B=2}$

$$\therefore \quad f(z) = \frac{-1}{(z-1)} + \frac{2}{(z-2)}$$

(i) Given region is $|z|<1$

$\Rightarrow \quad |z| < 1 < 2$

$\Rightarrow \quad |z|<1$ and $\left|\frac{z}{2}\right|<1$

$$f(z) = \frac{-1}{-(1-z)} + \frac{2}{-2(1-z/2)}$$

$$= (1-z)^{-1} - (1-z/2)^{-1}$$

$$= \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} (z/2)^n$$

(ii) Given region is $1 < |z| < 2$.

$$\Rightarrow \left|\frac{1}{z}\right| < 1 \quad \& \quad \left|\frac{z}{2}\right| < 1$$

$$\therefore f(z) = -1/z \left[\frac{1}{1-1/z}\right] + \frac{1}{(-2)} \frac{2}{(1-z/2)}$$

$$= -1/z \, (1-1/z)^{-1} - 2/2 \, (1-z/2)^{-1}$$

$$= -1/z \sum_{n=0}^{\infty} (1/z)^n - \sum_{n=0}^{\infty} (z/2)^n$$

(iii) Given region is $|z| > 2$

$$\Rightarrow \left|\frac{2}{z}\right| < 1$$

Also $\left|\frac{1}{z}\right| < 1$.

$$\therefore f(z) = -1/z \left(\frac{1}{1-1/z}\right) + 2/z \left(\frac{1}{1-2/z}\right)$$

$$= -1/z \, (1-1/z)^{-1} + 2/z \, (1-2/z)^{-1}$$

$$= -1/z \sum_{n=0}^{\infty} (1/z)^n + 2/z \sum_{n=0}^{\infty} (2/z)^n$$

Unit - V

Differential Equations

To find the complementary functions

Roots of Auxiliary Equation (A.E)	Complementary Function (C.F)
1) Roots are real & different m_1, m_2 $(m_1 \neq m_2)$	$Ae^{m_1 x} + Be^{m_2 x}$
2) Roots are real & equal $m_1 = m_2 = m$ (say)	$(Ax+B)e^{mx}$
3) Roots are imaginary $(\alpha \pm i\beta)$	$e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

To find Particular integral (P.I)

$$P.I = \frac{1}{f(D)} \cdot X$$

X	P.I
1) e^{ax}	$P.I = \frac{1}{f(D)} e^{ax} = e^{ax} \cdot \frac{1}{f(a)}$ if $f(a) \neq 0$. $= xe^{ax} \cdot \frac{1}{f'(a)}$ if $f(a)=0, f'(a)\neq 0$ $= x^2 e^{ax} \cdot \frac{1}{f''(a)}$ if $f(a)=0, f'(a)=0, f''(a)\neq 0$
2) $\sin ax$ (or) $\cos ax$	$P.I = \frac{1}{f(D)} [\cos ax \text{ (or) } \sin ax]$ Replace D^2 by $-a^2$
3) x^n	$P.I = \frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$ Expand $[f(D)]^{-1}$ & then operate.
4) $e^{ax} \cdot \phi(x)$	$P.I = \frac{1}{f(D)} e^{ax} \cdot \phi(x) = e^{ax} \cdot \frac{1}{f(D+a)} \phi(x)$

Problems

1) Solve $(D^2+6D+9)y=0$.

Soln

A.E is $m^2+6m+9=0$

$(m+3)(m+3)=0$

$\Rightarrow m=-3,-3$

Here, the roots are real & equal

$\therefore C.F = (Ax+B)e^{mx}$

$C.F = (Ax+B)e^{-3x}$

$$\boxed{y = C.F + P.I}$$

$y = (Ax+B)e^{-3x}$ $(\because P.I = 0)$

2) Solve $(D^2+1)y=0$ given $y(0)=0$, $y'(0)=1$

soln

Given $(D^2+1)y=0$

A.E is $m^2+1=0 \Rightarrow m^2=-1$

$m=\sqrt{-1}=\pm i$

ie) $m=0\pm 1i$

Here $\alpha=0$, $\beta=1$ (Compare $0\pm 1i$ with $\alpha\pm i\beta$)

$\therefore C.F = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

$= e^{0x}[A\cos x + B\sin x]$

$C.F = A\cos x + B\sin x$ $(\because e^0=1)$

$$\boxed{y = C.F + P.I}$$

$\therefore y = A\cos x + B\sin x$ $(\because P.I = 0)$

ie) $y(x) = A\cos x + B\sin x$ —— (1)

Given $y(0) = 0$

$\Rightarrow A\cos 0 + B\sin 0 = 0$

$\Rightarrow \boxed{A=0}$ $(\because \cos 0 = 1$

$y'(x) = -A\sin x + B\cos x$

Given $y'(0) = 1$

$\Rightarrow -A\sin 0 + B\cos 0 = 1$

$\Rightarrow \boxed{B = 1}$ ($\because \sin 0 = 0$, $\cos 0 = 1$)

$\therefore$ Eqn ① $\Rightarrow y(x) = 0.\cos x + 1.\sin x$

i.e) $y(x) = \sin x$ //.

3. Solve $(D^2 + 2D + 2)y = e^{-2x} + \cos 2x$.

Soln.

Given $(D^2 + 2D + 2)y = e^{-2x} + \cos 2x$

Let $(D^2 + 2D + 2)y = 0$

A.E is $m^2 + 2m + 2 = 0$

Here $a = 1,\ b = 2,\ c = 2$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{(2)^2 - 4\times1\times2}}{2\times1}$$

$$= \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2} = \frac{\cancel{2}(-1 \pm i)}{\cancel{2}}$$

$$= -1 \pm i$$

Here $\alpha = -1,\ \beta = 1$

$\therefore$ C.F $= e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

C.F $= e^{-x}[A\cos x + B\sin x]$

$$P.I_1 = \frac{1}{f(D)} e^{-2x}$$

$$= \frac{1}{D^2+2D+2} e^{-2x}$$

$$= \frac{1}{(-2)^2+2(-2)+2} e^{-2x} \quad (\text{Replace } D \text{ by } -2)$$

$$= \frac{1}{4-4+2} e^{-2x}$$

$$P.I_1 = \frac{1}{2} e^{-2x}$$

$$P.I_2 = \frac{1}{D^2+2D+2} \cos 2x$$

$$= \frac{1}{-4+2D+2} \cos 2x \quad [\text{Put } D^2 = -a^2 = -(2)^2 = -4]$$

$$= \frac{1}{2D-2} \cos 2x$$

$$= \frac{1}{(2D-2)} \frac{(2D+2)}{(2D+2)} \cos 2x.$$

$$= \frac{2D+2}{4D^2 - \cancel{4D} + \cancel{4D} - 4} \cos 2x$$

$$= \frac{2D+2}{4D^2-4} \cos 2x = \frac{\cancel{2}(D+1)}{\cancel{4}_{2}(D^2-1)} \cos 2x$$

$$= \frac{(D+1)}{2(D^2-1)} \cos 2x$$

$$= \frac{(D+1)}{2(-4-1)} \cos 2x \quad [\text{Put } D^2 = -4]$$

$$= \frac{(D+1)}{2 \times (-5)} \cos 2x$$

$$= \frac{-1}{10} [D(\cos 2x) + \cos 2x]$$

$$= \frac{-1}{10} [-2\sin 2x + \cos 2x]$$

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$$= \frac{2}{10}\sin 2x - \frac{1}{10}\cos 2x$$

$$P.I_2 = \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x$$

$$\boxed{y = C.F + P.I_1 + P.I_2}$$

$$y = e^{-x}\left[A\cos x + B\sin x\right] + \frac{1}{2}e^{-2x} + \frac{1}{5}\sin 2x - \frac{1}{10}\cos 2x$$

4) Solve $(D^2 - 3D + 2)y = 2\cos(2x+3) + 2e^{x}$

Soln: Given $(D^2 - 3D + 2)y = 2\cos(2x+3) + 2e^{x}$

A.E is $m^2 - 3m + 2 = 0$

$$(m-1)(m-2) = 0$$

$$m = 1, 2.$$

$$C.F = Ae^{m_1 x} + Be^{m_2 x}$$

$$= Ae^{1.x} + Be^{2.x}$$

$$C.F = Ae^{x} + Be^{2x}$$

$$P.I = \frac{1}{(D^2 - 3D + 2)}\left[2e^{x} + 2\cos(2x+3)\right]$$

$$= \frac{1}{D^2 - 3D + 2}(2e^{x}) + \frac{1}{(D^2 - 3D + 2)}2\cos(2x+3)$$

$$P.I_1 = \frac{1}{D^2 - 3D + 2} \cdot 2e^{x}$$

$$= 2 \cdot \frac{1}{(1)^2 - 3(1) + 2} \cdot e^{x} \quad [\text{Put } D = 1]$$

$$= 2 \cdot \frac{1}{1 - 3 + 2} \cdot e^{x}$$

$$= 2 \cdot \frac{1}{0} \cdot e^{x}$$

$$= \frac{2 \cdot xe^{x}}{2D - 3} = \frac{2x \cdot e^{x}}{2.1 - 3} \quad [\text{Put } D = 1]$$

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$$= \frac{2x.e^{x}}{-1}$$

$$P.I_1 = -2xe^{x}$$

$$P.I_2 = \frac{1}{D^2-3D+2} \times 2\cos(2x+3)$$

Formula

$$e^{ix} = \cos x + i\sin x$$

$$e^{i(2x+3)} = \cos(2x+3) + i\sin(2x+3)$$

$$= R.P \frac{2}{D^2-3D+2} e^{3i} . e^{i2x}$$

$$= R.P\ 2e^{3i} \times \frac{1}{D^2-3D+2} e^{i2x}$$

$$= R.P \frac{2e^{3i}.e^{i2x}}{(i2)^2-3(i2)+2} \quad [\text{Put } D = i2]$$

$$= R.P \frac{2e^{3i}.e^{i2x}}{i^2 2^2 - 6i+2}$$

$$= R.P \frac{2e^{i(3+2x)}}{(-1)4-6i+2}$$

$$= R.P \frac{2e^{i(3+2x)}}{-4-6i+2}$$

$$= R.P \frac{2e^{i(3+2x)}}{(-2-6i)} = R.P \frac{\cancel{2}e^{i(3+2x)}}{\cancel{2}(-1-3i)}$$

$$= R.P \frac{e^{i(3+2x)}}{-(1+3i)}$$

$$= R.P \frac{(1-3i)}{+(1-3i)} \times \frac{e^{i(3+2x)}}{-(1+3i)}$$

$$= R.P \frac{(1-3i)e^{i(3+2x)}}{-(1-3i)(1+3i)}$$

$$= R.P \frac{(1-3i)e^{i(3+2x)}}{-[1+\cancel{3i}-\cancel{3i}-9i^2]} = R.P \frac{(1-3i)e^{i(3+2x)}}{-[1+9]}$$

$$= R.P \frac{(1-3i)}{-10}\left[\cos(3+2x)+i\sin(3+2x)\right]$$

$$= R.P \frac{-1}{10}\left[\cos(3+2x)+i\sin(3+2x)-3i\cos(3+2x) -3i.i\sin(3+2x)\right]$$

$$= R.P \frac{-1}{10}\left[\cos(3+2x)+i\sin(3+2x)-3i\cos(3+2x) +3\sin(3+2x)\right] \quad (\because i^2=-1)$$

$$= \frac{-1}{10}\left[\cos(3+2x)+3\sin(3+2x)\right]$$

$$P.I_2 = \frac{-1}{10}\cos(3+2x) - \frac{3}{10}\sin(3+2x)$$

$$y = C.F + P.I_1 + P.I_2 \qquad (D^2+5D+6)y = 4\cos 5x.$$

$$y = e^x\left[A\cos x + B\sin x\right] - 2xe^x \frac{-1}{10}\cos(3+2x) + \frac{-3}{10}\sin(3+2x)$$

Formulae : $\quad \frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y = 8e^{x}-8e^{2x} \quad y(0)=2, y'(0)=3$

1) $(1+x)^{-1} = 1-x+x^2-x^3+\cdots$

2) $(1-x)^{-1} = 1+x+x^2+x^3+\cdots$

3) $(1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots$

4) $(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots$

5) Solve $(D^2+3D+2)y = \sin x + x^2$ (or) $y''+3y'+2y = \sin x + x^2$

Soln

A.E is $m^2+3m+2=0$.

$(m+1)(m+2)=0$

$\Rightarrow m = -1, -2.$

$C.F = Ae^{m_1 x}+Be^{m_2 x}$

$C.F = Ae^{-x}+Be^{-2x}$

$$P.I_1 = \frac{1}{(D^2+3D+2)}\sin x$$

$$= \frac{1}{(-1+3D+2)} \sin x \quad [\text{Put } D^2 = -a^2 = -1]$$

$$= \frac{1}{(3D+1)} \sin x = \frac{1}{(3D+1)} \times \frac{(3D-1)}{(3D-1)} \sin x$$

$$= \frac{(3D-1)\sin x}{(3D)^2 - (1)^2}$$

$$= \frac{(3D-1)\sin x}{9D^2 - 1}$$

$$= \frac{(3D-1)\sin x}{-9-1} \quad (\text{Put } D^2 = -a^2 = -1)$$

$$= \frac{(3D-1)\sin x}{-10}$$

$$= \frac{-1}{10}\left[3D(\sin x) - \sin x\right]$$

$$= \frac{-1}{10}\left[3\cos x - \sin x\right]$$

$$P.I_1 = \frac{-3}{10}\cos x + \frac{1}{10}\sin x$$

$$P.I_2 = \frac{1}{(D^2+3D+2)} \cdot x^2$$

$$= \frac{1}{2\left[1 + \left(\frac{D^2+3D}{2}\right)\right]} \cdot x^2$$

$$= \frac{1}{2} \cdot \left[1 + \left(\frac{D^2+3D}{2}\right)\right]^{-1} x^2$$

$$= \frac{1}{2} \cdot \left[1 - \left(\frac{D^2+3D}{2}\right) + \left(\frac{D^2+3D}{2}\right)^2 - \cdots\right] x^2$$

$$\left[\because (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\right]$$

$$= \frac{1}{2}\left[1 - \left(\frac{D^2+3D}{2}\right) + \left(\frac{\cancel{D^4} + 9D^2 + \cancel{6D^3}}{4}\right) - \cancel{\cdot}\cancel{\cdot}\right] x^2$$

$$(\because D(x^2) = 2x$$
$$D^2(x^2) = 2$$
$$D^3(x^2) = 0)$$

$$= \frac{1}{2}\left[1 - \frac{D^2}{2} - \frac{3D}{2} + \frac{9D^2}{4}\right] x^2$$

$$= \frac{1}{2}\left[x^2 - \frac{D^2(x^2)}{2} - \frac{3D(x^2)}{2} + \frac{9D^2(x^2)}{4}\right]$$

$$= \frac{1}{2}\left[x^2 - \frac{\cancel{2}}{\cancel{2}} - \frac{\overset{3}{\cancel{6}}x}{\cancel{2}} + \frac{\overset{9}{\cancel{18}}}{\underset{2}{\cancel{4}}}\right]$$

$$= \frac{1}{2}\left[x^2 - 1 - 3x + \frac{9}{2}\right]$$

$$= \frac{x^2}{2} - \frac{1}{2} - \frac{3x}{2} + \frac{9}{4}$$

$$= \frac{x^2}{2} - \frac{3x}{2} + \frac{9}{4} - \frac{1}{2}$$

$$= \frac{x^2}{2} - \frac{3x}{2} + \frac{9-2}{4}$$

$$P.I_2 = \frac{x^2}{2} - \frac{3x}{2} + \frac{7}{4}$$

$$\boxed{y = C.F + P.I_1 + P.I_2}$$

$$y = Ae^{-x} + Be^{-2x} - \frac{3}{10}\cos x + \frac{1}{10}\sin x + \frac{x^2}{2} - \frac{3x}{2} + \frac{7}{4} \,//$$

Problems based on R.H.S $= e^{ax}\phi(x)$

6) Solve $(D^2+5D+4)y = e^{-x}\sin 2x$.

soln: A.E is $m^2+5m+4=0$.

$$(m+1)(m+4)=0$$

$$m = -1, -4$$

$$C.F = Ae^{m_1 x} + Be^{m_2 x}$$

$$C.F = Ae^{-x} + Be^{-4x}$$

$$P.I = \frac{1}{(D^2+5D+4)} e^{-x} \sin 2x$$

$$= e^{-x} \cdot \frac{1}{[(D-1)^2+5(D-1)+4]} \sin 2x \quad \text{[Replace D by D+a ie) D-1]}$$

$$= e^{-x} \cdot \frac{1}{[D^2+1-2D+5D-5+4]} \sin 2x$$

$$= e^{-x} \cdot \frac{1}{[D^2+3D]} \sin 2x$$

$$= e^{-x} \cdot \frac{1}{[-4+3D]} \sin 2x \quad [\text{Put } D^2 = -a^2 = -(2)^2 = -4]$$

$$= e^{-x} \frac{1}{[3D-4]} \sin 2x$$

$$= e^{-x} \cdot \frac{1}{(3D-4)} \cdot \frac{(3D+4)}{(3D+4)} \sin 2x$$

$$= e^{-x} \frac{(3D+4)\sin 2x}{(3D)^2-(4)^2}$$

$$= e^{-x} \frac{(3D+4)\sin 2x}{9D^2-16}$$

$$= e^{-x} \frac{(3D+4)\sin 2x}{(9\times -4)-16}$$

$$= e^{-x} \frac{(3D+4)\sin 2x}{-36-16}$$

$$= e^{-x} \frac{[3D(\sin 2x)+4\sin 2x]}{-52}$$

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$$= \frac{e^{-x}[3\times 2\cos 2x + 4\sin 2x]}{-52}$$

$$= \frac{e^{-x}\times \cancel{2}[3\cos 2x + 2\sin 2x]}{-\cancel{52}\ 26}$$

$$= -\frac{3e^{-x}\cos 2x}{26} - \frac{\cancel{2}e^{-x}\sin 2x}{\cancel{26}\ 13}$$

$$P.I = \frac{-3e^{-x}}{26}\cos 2x - \frac{e^{-x}\sin 2x}{13}$$

$$y = C.F + P.I.$$

$$\text{ie) } y = Ae^{-x} + Be^{-4x} - \frac{3e^{-x}}{26}\cos 2x - \frac{e^{-x}\sin 2x}{13}$$ //.

7) Solve $(D^2 - 2D + 1)y = xe^x \sin x$

Soln A.E is $m^2 - 2m + 1 = 0$.

$$(m-1)(m-1) = 0$$

$$m = 1, 1$$

$$C.F = (Ax+B)e^{mx}$$

$$C.F = (Ax+B)e^{x}$$

$$P.I = \frac{1}{D^2 - 2D + 1} \cdot xe^x \sin x$$

$$= I.P \frac{1}{D^2 - 2D + 1} xe^x e^{ix}$$

$$= I.P \frac{1}{(D-1)^2} xe^{(1+i)x}$$

$$= I.P\ e^{(1+i)x} \cdot \frac{1}{(D+\cancel{1}+i-\cancel{1})^2} x \quad [\text{Put } D \to D+1+i]$$

$$= I.P\ e^{(1+i)x} \cdot \frac{1}{(D+i)^2} \cdot x$$

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$$= I.P\ e^{(1+i)x} \cdot \frac{1}{i^2\left[\frac{D}{i}+1\right]^2} \cdot x$$

$$= I.P\ e^{(1+i)x} \cdot \frac{1}{-1\left[1+\frac{D}{i}\right]^2} \cdot x$$

$$= I.P\ -e^{(1+i)x} \left[1+\frac{D}{i}\right]^{-2} \cdot x.$$

$$= I.P\ -e^{(1+i)x} \left[1 - \frac{2D}{i} + \frac{3D^2}{i^2} - \cdots\right] x \quad \left[\because D(x)=1,\ D^2(x)=0\right]$$

$$= I.P\ -e^{(1+i)x} \left[1 - \frac{2D}{i}\right] x$$

$$= I.P\ -e^{(1+i)x} \left[x - \frac{2D(x)}{i}\right]$$

$$= I.P\ -e^{(1+i)x} \left[x - \frac{2}{i}\right] \quad [\because D(x)=1]$$

$$= I.P\ -e^{(1+i)x} \left[x - \frac{2}{i} \times \frac{i}{i}\right]$$

$$= I.P\ -e^{(1+i)x} \left[x - \frac{2i}{-1}\right] \quad [\because i^2=-1]$$

$$= I.P\ -e^{(1+i)x} [x+2i]$$

$$= I.P\ -e^{x} \cdot e^{ix} [x+2i]$$

$$= I.P\ -e^{x} [\cos x + i\sin x][x+2i]$$

$$= I.P\ -e^{x} [x\cos x + ix\sin x + 2i\cos x - 2\sin x] \quad (\because i^2=-1)$$

$$P.I = -xe^{x}\sin x - 2e^{x}\cos x$$

$$y = C.F + P.I$$

$$y = (Ax+B)e^{x} - xe^{x}\sin x - 2e^{x}\cos x \ //.$$

Method of Variation of Parameter :-

Rule:-

Given $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X$

1) Find $C.F = Af_1 + Bf_2$

$A, B \rightarrow$ constants

$f_1, f_2 \rightarrow$ Function of x.

2) Find $w = f_1 f_2' - f_1' f_2$

3) $P.I = Pf_1 + Qf_2$

where $P = -\int \frac{f_2 X}{w} \, dx$

$Q = \int \frac{f_1 X}{w} \, dx.$

4) $y = C.F + P.I.$

Problems

① Solve $(D^2+a^2)y = \tan ax$ by method of variation of parameter.

Soln:-

Given $(D^2+a^2)y = \tan ax$.

A.E is $m^2 + a^2 = 0$.

$m^2 = -a^2$

$m = \pm ia$.

$m = 0 \pm ia$.

$\alpha = 0$, $\beta = a$.

$$C.F = e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$$

$$= e^{0x}\left[A\cos ax + B\sin ax\right]$$

$$C.F = A\cos ax + B.\sin ax.$$

$f_1 = \cos ax$ $\qquad$ $f_2 = \sin ax$

$f_1' = -a\sin ax$ $\qquad$ $f_2' = a\cos ax.$

$$w = f_1 f_2' - f_1' f_2$$

$$= a\cos^2 ax + a\sin^2 ax$$

$$w = a.$$

$$P.I = P.f_1 + Q.f_2.$$

$$P = -\int \frac{f_2 X}{w}\, dx$$

$$= -\int \frac{\sin ax \times \tan ax}{a}\, dx$$

$$= -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax}\, dx$$

$$= -\frac{1}{a}\int \left(\frac{1-\cos^2 ax}{\cos ax}\right) dx$$

$$= -1/a \int\left(\frac{1}{\cos ax} - \cos ax\right) dx$$

$$= -\frac{1}{a}\left[\int \sec ax \, dx - \int \cos ax \, dx\right]$$

$$= -\frac{1}{a}\left[\frac{1}{a}\log(\sec ax + \tan ax) - \frac{\sin ax}{a}\right]$$

$$= \frac{\sin ax}{a^2} - \frac{1}{a^2}\log(\sec ax + \tan ax).$$

$$Q = \int \frac{f_1 X}{\omega} dx$$

$$= \int \frac{\cos ax \times \tan ax}{a} dx$$

$$= \frac{1}{a}\int \sin ax . dx$$

$$= \frac{1}{a}\left(\frac{-\cos ax}{a}\right)$$

$$= \frac{-\cos ax}{a^2}$$

$$P.I = \left[\frac{\sin ax}{a^2} - \frac{1}{a^2}\log(\sec ax + \tan ax)\right]\cos ax$$

$$+ \left[\frac{-\cos ax}{a^2}\right]\sin ax$$

$$P.I = -\frac{1}{a^2}\log\left(\sec ax + \tan ax\right).$$

$$\therefore y = A\cos ax + B\sin ax - \frac{1}{a^2}\log(\sec ax + \tan ax).$$

(2) Solve $(D^2+a^2)y = \sec ax$ by method of Variation of parameter.

Soln:-

Given $(D^2+a^2)y = \sec ax$.

A.E is $m^2+a^2=0$.

$m^2 = -a^2$

$m = \pm ia$

$m = 0 \pm ia$.

$\alpha = 0$, $\beta = a$.

$\therefore$ C.F $= e^{0x}\left[A\cos ax + B\sin ax\right]$

C.F $= A\cos ax + B\sin ax$.

$f_1 = \cos ax$ $\qquad f_2 = \sin ax$.

$f_1' = -a\sin ax$ $\qquad f_2' = a\cos ax$

$w = f_1 f_2' - f_1' f_2$

$w = a\cos^2 ax + a\sin^2 ax$

$\boxed{w = a}$.

$P.I = P.f_1 + Q.f_2$.

$P = -\int \frac{f_2 X}{w}\, dx$.

$$P = -\int \frac{\sin ax}{a} \sec ax \,.\, dx$$

$$= -\frac{1}{a} \cdot \int \frac{\sin ax}{\cos ax} \,.\, dx$$

$$= -\frac{1}{a} \int \tan ax \,.\, dx$$

$$= -\frac{1}{a} \int \frac{a \,.\, \sin ax}{a \cos ax}$$

$$= \frac{1}{a^2} \int \frac{-a \sin ax}{\cos ax} \,.\, dx$$

$$P. = \frac{1}{a^2} \log (\cos ax).$$

$$Q = \int \frac{f_1 X}{w} \, dx$$

$$= \int \frac{\cos ax \times \sec ax}{a} \,.\, dx$$

$$= \frac{1}{a} \int dx$$

$$= \frac{x}{a}.$$

$$\therefore P.I = P.f_1 + Q f_2$$

$$= \frac{1}{a^2} \log (\cos ax) \,.\, \cos ax + \frac{x}{a} \sin ax$$

$$P.I = \frac{\cos ax}{a^2} \log \cos ax + \frac{x}{a} \sin ax.$$

$$y = A \cos ax + B \sin ax + \frac{\cos ax}{a^2} \log (\cos ax) + \frac{x}{a} \sin ax.$$

③ Solve $\left(\frac{d^2y}{dx^2}+y\right) = x\sin x$ by method of variation of parameter.

Soln:-

Given $\left(\frac{d^2y}{dx^2}+y\right) = x\sin x$.

i.e) $(D^2+1)y = x\sin x$.

A.E is $m^2+1=0$.

$m^2=-1$

$m=\pm i$

$m = 0\pm i$.

$\alpha=0,\ \beta=1$.

$\therefore$ C.F $= e^{0x}\left[A\cos x + B\sin x\right]$.

C.F $= A\cos x + B\sin x$.

$f_1 = \cos x$ $\qquad$ $f_2=\sin x$

$f_1' = -\sin x$ $\qquad$ $f_2' = \cos x$.

$w = f_1 f_2' - f_1' f_2$

$= \cos^2 x + \sin^2 x$

$\boxed{w=1.}$

$$P.I = Pf_1 + Qf_2$$

$$P = -\int \frac{f_2 \times X}{W} \, dx$$

$$= -\int \frac{\sin x \times x \sin x}{1} \, dx$$

$$= -\int x \sin^2 x \, dx$$

$$= -\int x \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= -\frac{1}{2}\int x \, dx + \frac{1}{2}\int x \cos 2x \, dx$$

$$= -\frac{1}{2}\left[\frac{x^2}{2}\right] + \frac{1}{2}\left[\frac{x \cdot \sin 2x}{2} + \frac{\cos 2x}{4}\right]$$

$$P. = -\frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8}.$$

$$Q = \int \frac{f_1 \times X}{W} \, dx$$

$$= \int \frac{\cos x \times x \sin x}{1} \, dx$$

$$= \int x \sin x \cos x \, dx$$

$$= \int x \frac{\sin 2x}{2} \, dx$$

$$= \frac{1}{2}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4} \right]$$

$$Q = -\frac{x\cos 2x}{2} + \frac{\sin 2x}{8}.$$

$$y = P.I = Pf_1 + Qf_2$$

$$= \left[-\frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8} \right]\cos x$$

$$+ \left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{8} \right]\sin x.$$

$$\therefore y = A\cos x + B\sin x + \left[-\frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8} \right]\cos x$$

$$+ \left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{8} \right]\sin x.$$

(4) Apply method of variation of parameter to solve $(D^2+4)y = \cot 2x$.

Soln:-

Given $(D^2+4)y = \cot 2x$.

A.E is $m^2+4 = 0$.

$$m^2 = -4$$

$$m = \pm 2i$$

$$m = 0 \pm 2i.$$

$$\alpha = 0, \quad \beta = 2.$$

$$C.F = e^{0x}\left[A\cos 2x + B\sin 2x\right]$$

$$C.F = A\cos 2x + B\sin 2x.$$

$f_1 = \cos 2x$ $\qquad$ $f_2 = \sin 2x$

$f_1' = -2\sin 2x$ $\qquad$ $f_2' = 2\cos 2x.$

$$w = f_1 f_2' - f_1' f_2$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$\boxed{w = 2}$$

$$P.I = Pf_1 + Qf_2.$$

$$P = -\int \frac{f_2 X}{w}\,dx$$

$$= -\int \frac{\cot 2x}{2} \times \sin 2x \,.\, dx.$$

$$= -\frac{1}{2}\int \sin^2 2x \times \frac{\cos 2x}{\sin 2x} \,.\, dx$$

$$= -\tfrac{1}{2}\int \cos 2x \,.\, dx$$

$$= -\tfrac{1}{2}\left(\frac{\sin 2x}{2}\right)$$

$$= \frac{-\sin 2x}{4}.$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cancel{\sin} \cos 2x \times \frac{\cos 2x}{\sin 2x}}{2} dx$$

$$= \frac{1}{2} \int \frac{\cos^2 2x}{\sin 2x} dx.$$

$$= \frac{1}{2} \int \frac{1 - \sin^2 2x}{\sin 2x} dx$$

$$= \frac{1}{2} \int \left(\frac{1}{\sin 2x} - \sin 2x \right) dx$$

$$= \frac{1}{2} \left[\int \operatorname{cosec} 2x \cdot dx - \int \sin 2x \cdot dx \right]$$

$$= \frac{1}{2} \left[\frac{\log(\operatorname{cosec} 2x - \cot 2x)}{2} + \frac{\cos 2x}{2} \right]$$

$$P.I = P f_1 + Q f_2$$

$$= \frac{1}{4} \left[\log(\operatorname{cosec} 2x - \cot 2x) \cancel{+} \right] \sin 2x.$$

$$\therefore y = A \cos 2x + B \sin 2x + \frac{1}{4} \left[\log(\operatorname{cosec} 2x - \cot 2x) \right] \sin 2x$$

Euler Type (or Cauchy Type)

Rule:-

Given $x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = X.$

1) Put $x = e^z \Rightarrow$ $\boxed{\begin{array}{l} \log x = z \\ xD = D' \\ x^2D^2 = D'(D'-1) \end{array}}$

Here $D = \frac{d}{dx}$, $D' = \frac{d}{dz}$.

2) Proceed the problem with D' in place of D & z inplace of x.

Problems

① Solve $(x^2D^2 - xD + 1)y = \sin(\log x)$ by Euler method.

soln. Given $(x^2D^2 - xD + 1)y = \sin(\log x)$.

Put $x = e^z \Rightarrow$ $\boxed{\begin{array}{l} z = \log x. \\ x^2D^2 = D'(D'-1) \\ xD = D'. \end{array}}$

$$(D'(D'-1) - D' + 1)y = \sin z$$

$$(D'^2 - 2D' + 1)y = \sin z.$$

A.E is $m^2 - 2m + 1 = 0$.

$$(m-1)^2 = 0.$$

$$m = 1, 1.$$

$$\therefore \quad C.F = e^{z}\left[Az + B\right]$$

$$= e^{\log x}\left[A\log x + B\right]$$

$$C.F. = x\left[A\log x + B\right].$$

$$P.I = \frac{1}{(D'^2 - 2D' + 1)} \sin z$$

$$= \frac{1}{-1 - 2D' + 1} \sin z$$

$$= \frac{1}{-2D'} \sin z$$

$$= -\frac{1}{2}\int \sin z . dz$$

$$= -\frac{1}{2}\left(-\cos z\right)$$

$$= \frac{\cos z}{2}$$

$$P.I = \frac{\cos(\log x)}{2}$$

$$y = C.F + P.I$$

$$= \left(A\log x + B\right) + \frac{1}{2}\cos(\log x).$$

(2) Solve $\left(x^2D^2 - xD + 1\right)y = \left[\frac{\log x}{x}\right]^2$

Soln:-

Given $\left[x^2D^2 - xD + 1\right]y = \left[\frac{\log x}{x}\right]^2$

Put $x = e^z \Rightarrow$ $\boxed{\begin{array}{l} z = \log x \\ x^2D^2 = D'(D'-1) \\ xD = D' \end{array}}$

$$\therefore \left[D'(D'-1) - D' + 1\right]y = \left(\frac{z}{e^z}\right)^2$$

$$\left(D'^2 - 2D' + 1\right)y = e^{-2z} \cdot z^2.$$

A.E is $m^2 - 2m + 1 = 0.$

$m = 1, 1.$

$$\therefore C.F = e^z(Az + B)$$

$$= e^{\log x}(A\log x + B)$$

$$= x\,(A\log x + B)$$

$$P.I = \frac{1}{(D'^2 - 2D' + 1)} e^{-2z} \cdot z^2$$

$$= e^{-2z} \frac{1}{\left[(D'-2)^2 - 2(D'-2) + 1\right]} z^2.$$

$$= e^{-2z} \frac{1}{(D'^2 - 4D' + 4) - 2D' + 4 + 1} z^2$$

$$= e^{-2z} \frac{1}{(D'^2 - 6D' + 9)} z^2$$

$$= \frac{e^{-2z}}{9} \left(1 + \frac{D'^2 - 6D'}{9}\right)^{-1} z^2$$

$$= \frac{e^{-2z}}{9} \left[1 - \left(\frac{D'^2 - 6D'}{9}\right) + \left(\frac{D'^2 - 6D'}{9}\right)^2 - \cdots\right] z^2$$

$$= \frac{e^{-2z}}{9} \left[z^2 - \frac{(D'^2)(z^2)}{9} + \frac{6D'(z^2)}{9} + \frac{36}{81}(D')^2 z^2\right]$$

($\because$ neglecting D'^3 and higher power)

$$= \frac{e^{-2z}}{9} \left[z^2 - 2/9 + \frac{12z}{9} + \frac{72}{81}\right]$$

$$= \frac{e^{-2z}}{9} \left[z^2 + \frac{12z}{9} + \frac{72 - 18}{81}\right]$$

$$\text{P.I} = \frac{e^{-2z}}{9} \left[z^2 + \frac{4z}{3} + \frac{2}{3}\right]$$

$$y = C.F + P.I$$

$$= x(A\log x + B) + \frac{e^{-2z}}{9}\left[z^2 + \frac{4z}{3} + \frac{2}{3}\right]$$

$$= x(A\log x + B) + \frac{e^{-2\log x}}{9}\left[(\log x)^2 + \frac{4\log x}{3} + \frac{2}{3}\right]$$

$$y = x(A\log x + B) + \frac{1}{9x^2}\left[(\log x)^2 + \frac{4\log x}{3} + \frac{2}{3}\right]$$

③ Solve $(x^2D^2 + 4xD + 2)y = x\log x$.

Soln:-

Given $(x^2D^2 + 4xD + 2)y = x\log x$.

Put $x = e^z \Rightarrow$ $\boxed{\begin{array}{l} z = \log x \\ x^2D^2 = D'(D'-1) \\ xD = D' \end{array}}$

$$\therefore \left[D'(D'-1) + 4D' + 2\right]y = e^z z$$

$$\left[D'^2 + 3D' + 2\right]y = e^z . z.$$

A.E is

$$m^2 + 3m + 2 = 0.$$

$$(m+1)(m+2) = 0$$

$$m = -1, \quad m = -2.$$

$$C.F = Ae^{-z} + Be^{-2z}$$

$$= Ae^{-\log x} + Be^{-2\log x}$$

$$C.F = \frac{A}{x} + \frac{B}{x^2}$$

$$P.I = \frac{1}{(D'^2 + 3D' + 2)} e^{z} . z$$

$$= e^{z} . \frac{1}{\left[(D'+1)^2 + 3(D'+1) + 2\right]} . z$$

$$= e^{z} \frac{1}{(D'^2 + 2D' + 1 + 3D' + 3 + 2)} z$$

$$= e^{z} \frac{1}{(D'^2 + 5D' + 6)} z$$

$$= \frac{e^{z}}{6} \left(1 + \frac{D'^2 + 5D'}{6}\right)^{-1} z$$

$$= \frac{e^{z}}{6} \left[1 - \left(\frac{D'^2 + 5D'}{6}\right) + \left(\frac{D'^2 + 5D'}{6}\right)^2 - \cdots\right] z$$

$$= e^{z}/6 \left[z - \frac{5}{6} D'(z)\right]$$

$$= e^{z}/6 \left[z - 5/6\right]$$

$$P.I = \frac{e^z}{6}\left(z - \frac{5}{6}\right)$$

$$P.I = \frac{x}{6}\left(\log x - \frac{5}{6}\right)$$

$$\therefore \; y = C.F + P.I$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{x}{6}\left[\log x - \frac{5}{6}\right]$$

(4) Solve $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$

Soln:-

Given $(xD^2 + D)y = 0.$

Multiply by x.

$$(x^2D^2 + xD)y = 0$$

Put $x = e^z \Rightarrow$

$$\boxed{\begin{array}{l} z = \log x \\ x^2D^2 = D'(D'-1) \\ xD = D' \end{array}}$$

$$\therefore \; [D'(D'-1) + D']y = 0.$$

$$(D'^2)y = 0.$$

A.E is $m^2 = 0$

$m = 0, 0.$

$$\therefore \; C.F = e^{0z}(Az + B)$$

$$= A\log x + B.$$

P

$\therefore\ y = C.F$

$= Az + B$

$= A\log x + B.$

(5) Solve $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$.

Soln:

Given $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$.

Put $x = e^z \Rightarrow$

$$\boxed{\begin{array}{l} z = \log x \\ x^2D^2 = D'(D'-1) \\ xD = D' \end{array}}$$

$\therefore\ [D'(D'-1) - 2D' - 4]\,y = e^{2z} + 2z$

$[D'^2 - 3D' - 4]\,y = e^{2z} + 2z.$

A.E is

$m^2 - 3m - 4 = 0.$

-4	$+1$
m	m

$(m-4)(m+1) = 0.$

$m = 4,\ m = -1.$

$$\therefore \quad C.F = Ae^{4z} + Be^{-z}$$

$$= A e^{4\log x} + B e^{-\log x}$$

$$\cancel{= \frac{A}{x^4} +}$$

$$= A x^4 + \frac{B}{x}.$$

<u>To find P.I</u>

$$P.I = \frac{1}{(D'^2 - 3D' - 4)} (e^{2z} + 2z)$$

$$= P.I_1 + P.I_2 \longrightarrow (1)$$

where $\quad P.I_1 = \dfrac{1}{(D'^2 - 3D' - 4)} e^{2z}$

& $\quad P.I_2 = \dfrac{1}{(D'^2 - 3D' - 4)} 2z.$

$$P.I_1 = \frac{1}{(D'^2 - 3D' - 4)} e^{2z}.$$

$$= \frac{1}{4 - 6 - 4} e^{2z}$$

$$= -\frac{1}{6} e^{2z}$$

$$= -\frac{1}{6} e^{2\log x}$$

$$P.I_1 = -\frac{1}{6} x^2$$

$$P.I_2 = \frac{1}{(D'^2 - 3D' - 4)} \cdot 2z$$

$$= \frac{1}{-4}\left(1 + \left(\frac{D'^2 - 3D'}{-4}\right)\right)^{-1} 2z$$

$$= -\frac{1}{4}\left[1 - \left(\frac{D'^2 - 3D'}{-4}\right) + \left(\frac{D'^2 - 3D'}{-4}\right)^2 - \cdots\right] 2z$$

$$\cancel{= -\frac{1}{4}\left[z^2 + \frac{D'^2}{4}(z^2) - \frac{3D'}{4}(z^2) + \frac{9D'^2}{16}(z^2)\right]}$$

$$\cancel{= -\frac{1}{4}\left[z^2 + \frac{2}{4} - \frac{6z}{4} + \frac{9}{16}\right]}$$

$$\cancel{= -\frac{1}{4}\left[z^2\right.}$$

$$P.I_2 = -\frac{1}{4}\left[2z - \frac{3D'}{4}(2z)\right]$$

$$= -\frac{2}{4}\left[z - \frac{3}{4}\right]$$

$$= -\frac{1}{2}\left[z - \frac{3}{4}\right]$$

$$= -\frac{1}{2}\left[\log x - \frac{3}{4}\right]$$

$$y = C.F + P.I_1 + P.I_2$$

$$= Ax^4 + \frac{B}{x} - \frac{1}{6}x^2 - \frac{1}{2}\left[\log x - \frac{3}{4}\right]$$

(6) Solve $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$.

Soln:-

Given $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Multipey by x^2

$$(x^2D^2 + xD)y = 12\log x.$$

Put $x = e^z \Rightarrow$

$$\boxed{\begin{array}{l} z = \log x \\ x^2D^2 = D'(D'-1) \\ xD = D' \end{array}}$$

$\therefore$ $[D'(D'-1) + D']\, y = 12z.$

$$(D'^2)y = 12z.$$

A.E is $m^2 = 0.$

$m = 0, 0.$

$$C.F = e^{0z}[Az + B]$$

$$= A\log x + B.$$

To find P.I

$$P.I = \frac{1}{D'^2}(12z)$$

$$= 12 \frac{1}{D'^2} z$$

$$= 12 \iint z.dz.dz$$

$$= 12 \int \frac{z^2}{2} dz$$

$$= 6 \int z^2 dz$$

$$= 6 \frac{z^3}{3}$$

$$= 2z^3.$$

$$= 2(\log x)^3$$

$$\therefore \quad y = C.F + P.I$$

$$= A\log x + B + 2(\log x)^3.$$

Legendre Type

Given $(ax+b)^2 \frac{d^2y}{dx^2} + a_1 (ax+b)\frac{dy}{dx} + a_2 y = X$

Rules:-

1) Put $ax+b = e^z$

$\Rightarrow \quad z = \log(ax+b)$

$(ax+b)^2 D^2 = a^2 D'(D'-1)$

$(ax+b) D = a D'$

Here $D = \frac{d}{dx}$, $D' = \frac{d}{dz}$.

2) Proceed the problem with D' inplace of D and z inplace of x.

Problems:-

① Solve $\left[(x+1)^2 D^2 + (x+1) D + 1\right] y = 4 \cos\left[\log(x+1)\right]$.

Soln:

Given $\left[(x+1)^2 D^2 + (x+1) D + 1\right] y = 4 \cos\left[\log(x+1)\right]$.

Put $x+1 = e^z$

$z = \log(x+1)$

$(x+1)^2 D^2 = 1^2 D'(D'-1) = D'(D'-1)$

$(x+1) D = 1 D'$

$= D'$.

$$\therefore \left[D'(D'-1)+D'+1\right]y = 4\cos z.$$

$$\left[D'^2+1\right]y = 4\cos z.$$

A.E is

$$m^2+1=0.$$

$$m^2=-1.$$

$$m=0\pm i.$$

$$\alpha=0,\ \beta=1.$$

$$\therefore\ C.F = e^{\alpha z}\left[A\cos\beta z + B\sin\beta z\right]$$

$$= e^{0z}\left[A\cos z + B\sin z\right]$$

$$= A\cos z + B\sin z$$

$$C.F. = A\cos\left[\log(x+1)\right] + B\sin\left[\log(x+1)\right].$$

$$P.I = \frac{1}{D'^2+1}4\cos z$$

$$= 4\frac{1}{D'^2+1}\cos z$$

$$= 4\frac{1}{-1+1}\cos z$$

$$= 4\frac{1}{0}\cos z.$$

$$\therefore\ P.I = 4z\frac{1}{2D'}\cos z$$

$$= \frac{4z}{2}\frac{1}{D'}\cos z$$

$$= 2z \int \cos z \, dz$$

$$= 2z \cdot \sin z$$

$$P.I = 2\log[x+1] \sin[\log(x+1)]$$

$$y = C.F + P.I$$

$$y = A\cos[\log(x+1)] + B\sin[\log(x+1)]$$

$$+ 2\log(x+1) \cdot \sin[\log(x+1)]$$

(2) Solve $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1$.

Soln:-

Given

$$(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1.$$

$$(i.e.) \left[(3x+2)^2 D^2 + 3(3x+2)D - 36\right] y = 3x^2+4x+1.$$

Put $3x+2 = e^z$

$$\Rightarrow z = \log(3x+2)$$

$$(3x+2)^2 D^2 = 3^2 D'(D'-1) = 9D'(D'-1)$$

$$(3x+2) D = 3D'$$

$$x = \frac{e^z - 2}{3}$$

$$\therefore \left[9D'(D'-1)+9D'-36\right]y = 3\left(\frac{e^z-2}{3}\right)^2+4\left(\frac{e^z-2}{3}\right)+1$$

$$9\left[D'(D'-1)+D'-4\right]y = \frac{e^{2z}-4e^z+4+4e^z-8+3}{3}$$

$$\left(D'^2-4\right)y = \frac{e^{2z}-1}{9\times 3}$$

$$\left(D'^2-4\right)y = \frac{e^{2z}}{27} - \frac{1}{27}.$$

A.E is

$$m^2-4=0.$$

$$m^2=4.$$

$$m=\pm 2$$

$$m_1=2,\ m_2=-2.$$

$$C.F = Ae^{m_1 z}+Be^{m_2 z}$$

$$= Ae^{2z}+Be^{-2z}$$

$$= Ae^{2\log(3x+2)} + Be^{-2\log(3x+2)}$$

$$C.F = A(3x+2)^2 + B(3x+2)^{-2}$$

$$P.I = \frac{1}{(D'^2-4)}\left[\frac{e^{2z}}{27}-\frac{1}{27}\right]$$

$$= P.I_1 + P.I_2$$

where $P.I_1 = \frac{1}{(D'^2-4)}\left(\frac{e^{2z}}{27}\right)$

& $P.I_2 = \frac{1}{(D'^2-4)}\left(\frac{-1}{27}\right)$.

$$P.I_1 = \frac{1}{(D'^2-4)} \frac{e^{2z}}{27}.$$

$$= \frac{1}{27} \frac{1}{(D'^2-4)} e^{2z}$$

$$= \frac{1}{27}\left[\frac{1}{4-4}\right] e^{2z}$$

$$= \frac{1}{27}\left(\frac{1}{0}\right) e^{2z}$$

$$\therefore P.I_1 = \frac{z}{27}\left[\frac{1}{2D'}\right] e^{2z}$$

$$= \frac{z}{27}\left[\frac{1}{4}\right] e^{2z}$$

$$= \frac{z e^{2z}}{108}.$$

$$P.I_2 = \frac{1}{(D'^2-4)}\left(-\frac{1}{27}\right)$$

$$= -\frac{1}{27} \frac{1}{(D'^2-4)}(1)$$

$$= -\frac{1}{27}\frac{1}{(D'^2-4)}(e^{0z}) \qquad \left(\because e^{0z}=1\right)$$

$$= -\frac{1}{27}\left(\frac{1}{-4}\right)e^{0z}$$

$$= \frac{1}{108}e^{0z}$$

$$P.I_2 = \frac{1}{108}$$

$$\therefore y = C.F + P.I_1 + P.I_2$$

$$= A(3x+2)^2 + B(3x+2)^{-2} + \frac{ze^{2z}}{108} + \frac{1}{108}.$$

(3) Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin\left[\log(1+x)\right]$

Soln :-

Given

$$\left[(1+x)^2D^2 + (1+x)D + 1\right]y = 2\sin\left[\log(1+x)\right].$$

Put $1+x = e^z$

$$\Rightarrow z = \log(1+x)$$

$$(1+x)^2D^2 = 1^2D'(D'-1) = D'(D'-1)$$

$$(1+x)D = 1D' = D'.$$

$$\therefore \left[D'(D'-1) + D' + 1\right] y = 2\sin z$$

$$\left[D'^2 + 1\right] y = 2\sin z.$$

A.E is

$$m^2 + 1 = 0.$$

$$m^2 = -1.$$

$$m = 0 \pm i.$$

$$\alpha = 0, \quad \beta = 1.$$

$$C.F = e^{\alpha z}\left[A\cos\beta z + B\sin\beta z\right]$$

$$= e^{0z}\left[A\cos z + B\sin z\right]$$

$$= A\cos z + B\sin z$$

$$C.F = A\cos\left[\log(x+1)\right] + B\sin\left[\log(x+1)\right]$$

$$P.I = \frac{1}{(D'^2+1)} 2\sin z$$

$$= 2 \frac{1}{(D'^2+1)} \sin z$$

$$= 2 \frac{1}{-1+1} \sin z$$

$$= 2 \left(\frac{1}{0}\right) \sin z$$

$$\therefore \; P.I = 2z \cdot \frac{1}{2D'} (\sin z)$$

$$= z \cdot \frac{1}{D'} (\sin z)$$

$$= z \int \sin z \, dz$$

$$P.I. = -z \cos z$$

$$= -\log(x+1) \cdot \cos\left[\log(x+1)\right]$$

$$\therefore \; y = C.F + P.I$$

$$y = A \cos\left[\log(1+x)\right] + B \sin\left[\log(1+x)\right]$$
$$- \log(x+1) \cos\left[\log(x+1)\right].$$

(4) Transform the equation

$$(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x$$

into a linear differential eqn with constant coefficients.

Soln:-

Given

$$(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x.$$

Put $(2x+3) = e^z$

$$\Rightarrow \quad z = \log(2x+3)$$

$$(2x+3)^2 D^2 = 2^2 D'(D'-1)$$

$$= 4\ D'(D'-1)$$

$$(2x+3)\ D = 2D'.$$

$$\therefore \left[4D'(D'-1) - 2\times 2D' - 12\right] y = 6\left(\frac{e^z - 3}{2}\right)$$

$$4\left[D'^2 - D' - \cancel{4}D' - \cancel{12}\,3\right] y = 6\left(\frac{e^z - 3}{2}\right)$$

$$\left(D'^2 - 2D' - 3\right) y = \frac{6}{8}\left(e^z - 3\right)$$

$$\left(D'^2 - 2D' - 3\right) y = \frac{3}{4}\left(e^z - 3\right).$$

is required form.

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Simultaneous Differential Equation

Problems :-

① Solve the simultaneous diff. eqns

$$\frac{dx}{dt} + 2y = \sin 2t \quad \text{and}$$

$$\frac{dy}{dt} - 2x = \cos 2t.$$

Soln:-

Put $\frac{d}{dt} = D$.

$\therefore$ $Dx + 2y = \sin 2t$

$Dy - 2x = \cos 2t.$

(i) $Dx + 2y = \sin 2t \longrightarrow ①$

$-2x + Dy = \cos 2t \longrightarrow ②$

$① \times D - 2 \times ②$

$\Rightarrow$

$$D^2x + 2Dy = D(\sin 2t) = 2\cos 2t$$

$$-4x + 2Dy = 2\cos 2t.$$

(−)

$$D^2x + 4x = 0.$$

$$(D^2+4)x = 0.$$

Solve : $(D^2+4)x = 0.$

A.E is $m^2+4=0$.

$m^2=-4$.

$m=\pm i2$

$m=0\pm i2$.

$\alpha=0,\ \beta=2$.

$\therefore\ C.F = e^{\alpha t}\left[A\cos\beta t + B\sin\beta t\right]$

$C.F = e^{0t}\left[A\cos 2t + B\sin 2t\right]$

$C.F. = A\cos 2t + B\sin 2t$

$\therefore\ x(t) = A\cos 2t + B\sin 2t$

$$\frac{dx}{dt} = -2A\sin 2t + 2B\cos 2t.$$

Given : $\dfrac{dx}{dt} + 2y = \sin 2t$.

$$\Rightarrow\quad 2y = \sin 2t - \frac{dx}{dt}$$

$$2y = \sin 2t + 2A\sin 2t - 2B\cos 2t.$$

$$y = \frac{\sin 2t}{2} + A\sin 2t - B\cos 2t.$$

$$y(t) = \frac{\sin 2t}{2} + A\sin 2t - B\cos 2t.$$

$\therefore$ Solutions are

$$x(t) = A\cos 2t + B\sin 2t$$

$$y(t) = \frac{\sin 2t}{2} + A\sin 2t - B\cos 2t.$$

(2) Solve $\frac{dx}{dt} - y = t$ and

$\frac{dy}{dt} + x = t^2$ given $x(0) = y(0) = 2$.

Soln.

Given $\frac{dx}{dt} - y = t$

$\frac{dy}{dt} + x = t^2$

Put $\frac{d}{dt} = D$

$Dx - y = t$

$Dy + x = t^2$

(i) $Dx - y = t \longrightarrow (1)$

$x + Dy = t^2 \longrightarrow (2)$

$(1) \times D \Rightarrow D^2 x - Dy = D(t) = 1.$

$(2) \Rightarrow x + Dy = t^2.$

(+)

$D^2 x + x = 1 + t^2$

(i) $(D^2+1)x = 1+t^2.$

Solve : $(D^2+1)x = 1+t^2.$

A.E is $m^2+1 = 0.$

$m^2 = -1.$

$m = \pm i.$

$m = 0 \pm i.$

$\alpha = 0, \beta = 1.$

$\therefore \quad C.F = e^{\alpha t}\left[A\cos\beta t + B\sin\beta t\right]$

$= e^{0t}\left[A\cos t + B\sin t\right]$

$C.F = A\cos t + B\sin t$

To find P.I

$$P.I = \frac{1}{(D^2+1)}(1+t^2)$$

$$= (1+D^2)^{-1}(1+t^2)$$

$$= \left[1 - D^2 + D^4 - \dots\right]\left[1+t^2\right]$$

$$= (1+t^2) - D^2(1+t^2)$$

$$= (1+t^2) - \left[\frac{d}{dt}\left(\frac{d}{dt}(1+t^2)\right)\right]$$

$$= 1+t^2 - \left[\frac{d}{dt}(2t)\right]$$

$$= 1+t^2-2$$

$$= t^2-1.$$

$\therefore \quad x(t) = C.F + P.I$

$x(t) = A\cos t + B\sin t + t^2 - 1. \longrightarrow (3)$

$$\frac{dx}{dt} = -A\sin t + B\cos t + 2t$$

Given $\frac{dx}{dt} - y = t$

$$\Rightarrow y = \frac{dx}{dt} - t$$

$$y = -A\sin t + B\cos t + 2t - t$$

$$y(t) = -A\sin t + B\cos t + t \longrightarrow (4)$$

Also given that $x(0) = 2$

$\therefore$ Put $t=0$ in (3)

$$x(0) = A\cos 0 + B\sin 0 + 0^2 - 1$$

$$2 = A - 1$$

$$\Rightarrow \boxed{A = 3}$$

Put $t=0$ in (4)

$$y(0) = -A\sin 0 + B\cos 0 + 0$$

$$\boxed{2 = B}$$

Sub $A=3$ & $B=2$ in (3) & (4)

Solutions are

$$x(t) = 3\cos t + 2\sin t + t^2 - 1$$

$$y(t) = -3\sin t + 2\cos t + t.$$

(3) Solve the simultaneous diff eqns.

$$\frac{dx}{dt} + 2x + 3y = 2e^{2t}$$

$$\frac{dy}{dt} + 3x + 2y = 0.$$

Soln:-

Given $\frac{dx}{dt} + 2x + 3y = 2e^{2t}$

$\frac{dy}{dt} + 3x + 2y = 0.$

Put $\frac{d}{dt} = D.$

$$Dx + 2x + 3y = 2e^{2t}$$

$$Dy + 3x + 2y = 0.$$

$$\Rightarrow \quad (D+2)x + 3y = 2e^{2t} \longrightarrow (1)$$

$$3x + (D+2)y = 0. \longrightarrow (2).$$

$(1) \times (D+2) \Rightarrow (D+2)^2 x + 3(D+2)y = (D+2)2e^{2t}.$

$(2) \times 3 \Rightarrow 9x + 3(D+2)y = 0.$

$(-)$ $(D+2)^2 x - 9x = (D+2)(2e^{2t})$

$$(D^2+4D+4)x - 9x = D(2e^{2t}) + 4e^{2t}$$

$$(D^2+4D+4-9)x = 4e^{2t} + 4e^{2t}$$

$$(D^2+4D-5)x = 8e^{2t}.$$

Solve : $(D^2+4D-5)x = 8e^{2t}$.

A.E is $m^2+4m-5=0$.

$$\begin{array}{c|c} 5 & -1 \\ \hline m & m \end{array}$$

$$(m+5)(m-1)=0.$$

$$m=-5, \quad m=1.$$

$$m_1=-5, \quad m_2=1.$$

$$C.F = Ae^{m_1 t} + Be^{m_2 t}$$

$$= Ae^{-5t} + Be^{t}.$$

To find P.I :-

$$P.I = \frac{1}{(D^2+4D-5)} 8e^{2t}$$

$$= 8 \frac{1}{(D^2+4D-5)} e^{2t}$$

$$= 8 \frac{1}{2^2+8-5} e^{2t}$$

$$= \frac{8}{12+8-5} e^{2t}$$

$$= \frac{8}{7} e^{2t}$$

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$\therefore x(t) = C.F + P.I$

$$x(t) = Ae^{-5t} + Be^{t} + \frac{8}{7}e^{2t}$$

$$\frac{dx}{dt} = -5Ae^{-5t} + Be^{t} + \frac{16}{7}e^{2t}.$$

From given eqns

$$\frac{dx}{dt} + 2x + 3y = 2e^{2t}.$$

$$\therefore 3y = 2e^{2t} - \frac{dx}{dt} - 2x$$

$$= 2e^{2t} - \left(-5Ae^{-5t} + Be^{t} + \frac{16}{7}e^{2t}\right) - 2\left(Ae^{-5t} + Be^{t} + \frac{8}{7}e^{2t}\right)$$

$$= 2e^{2t} + 5Ae^{-5t} - Be^{t} - \frac{16}{7}e^{2t} - 2Ae^{-5t} - 2Be^{t} - \frac{16}{7}e^{2t}.$$

$$3y = 3Ae^{-5t} - 3Be^{t} - \left(\frac{32}{7} - 2\right)e^{2t}$$

$$= 3Ae^{-5t} - 3Be^{t} - \left(\frac{32-14}{7}\right)e^{2t}$$

$$= 3Ae^{-5t} - 3Be^{t} - \left(\frac{18}{7}\right)e^{2t}.$$

$$\therefore y(t) = Ae^{-5t} - Be^{t} - \frac{6}{7}e^{2t}.$$

$\therefore$ Solutions are

$$x(t) = Ae^{-5t} + Be^{t} + \frac{8}{7} e^{2t}$$

$$y(t) = Ae^{-5t} - Be^{t} - \frac{6}{7} e^{2t}$$

(4) Solve the simultaneous diff eqns

$$\frac{dx}{dt} + 2y = -\sin t$$

$$\frac{dy}{dt} - 2x = \cos t \quad \text{given } x(0)=1, \ y(0)=0.$$

Soln:-

Put $\frac{d}{dt} = D$.

Given $Dx + 2y = -\sin t$

$Dy - 2x = \cos t$.

(ie) $Dx + 2y = -\sin t \longrightarrow (1)$

$-2x + Dy = \cos t. \longrightarrow (2)$

$(1) \times D \Rightarrow \quad D^2x + 2Dy = D(-\sin t) = -\cos t$

$(2) \times 2 \Rightarrow \quad -4x + 2Dy = 2\cos t$

$(-) \quad D^2x + 4x = -3\cos t.$

$$(D^2+4)x = -3\cos t$$

Solve : $(D^2+4)x = -3\cos t$

A.E is $m^2+4=0$.

$m^2 = -4$.

$m = \pm i2$.

$m = 0 \pm i2$.

$\alpha = 0, \quad \beta = 2$.

$C.F = e^{\alpha t}\left[A\cos\beta t + B\sin\beta t\right]$

$= e^{0t}\left[A\cos 2t + B\sin 2t\right]$

$C.F. = A\cos 2t + B\sin 2t$.

To find. P.I

$P.I = \dfrac{1}{(D^2+4)}(-3\cos t)$

$= -3\dfrac{1}{(D^2+4)}\cos t$

$= -3\dfrac{1}{-1+4}\cos t$

$= -3\dfrac{1}{3}\cos t$

$= -\cos t$.

$\therefore \quad x(t) = C.F + P.I.$